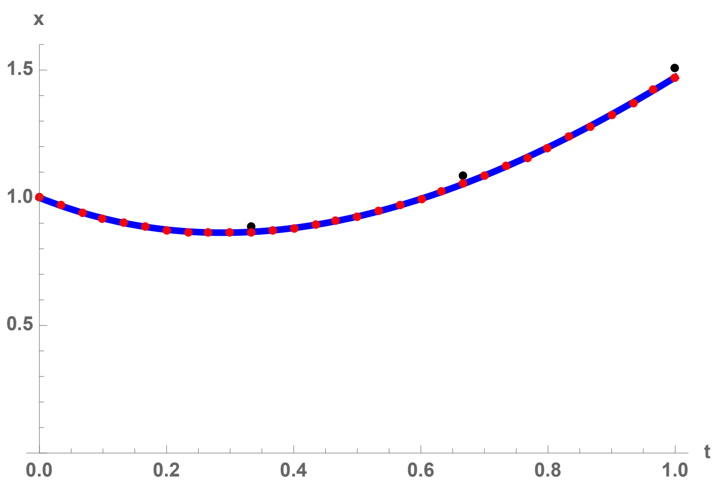
Math 376 Homework Set #5 Solutions

1. (5 points) For the IVP $x' = -x + 3t$, $x(0) = 1$, use Improved Euler's Method with $h = \frac{1}{3}$ and with $h = \frac{1}{30}$ to approximate $x(1)$. Plot both your approximations as well as the actual solutions, and calculate the error between these approximations and the actual solution at $t = 1$. (Note: this is an IVP from a previous homework!) What do you expect the error to be if you used $h = \frac{1}{300}$?

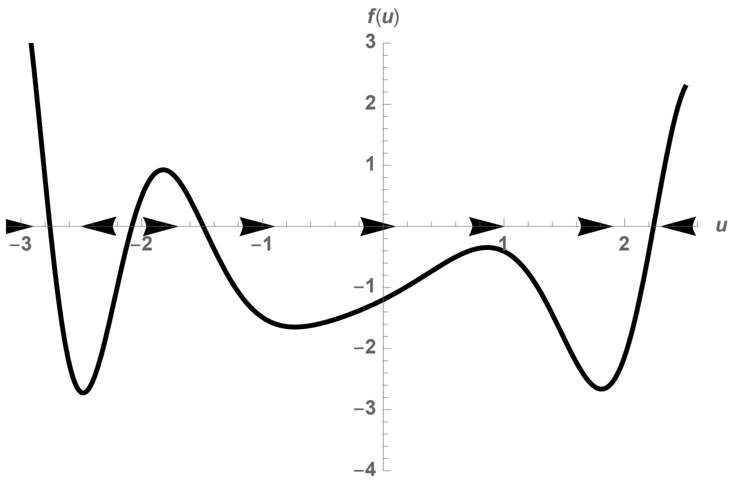
   The actual solution of the IVP is $x(t) = 3t - 3 + 4e^{-t}$. With $h = \frac{1}{3}$, the Improved Euler Method gives an approximation of $x(1) \approx \frac{2197}{1458}$, and so the error with $h = \frac{1}{3}$ is $\left|\frac{4}{e} - \frac{2197}{1458}\right| \approx .0353409$. With $h = \frac{1}{300}$, the Improved Euler Method gives and approximation of $x(1) \approx 1.4718$, and so the error with $h = \frac{1}{30}$ is $\left|\frac{4}{e} - 1.4718\right| \approx .000279433$. (Notice that we gained two decimal places, as we'd expect since going from $h = \frac{1}{3}$ to $h = \frac{1}{30}$ means multiplying h by $\frac{1}{10}$ and so the error should get multiplied by $\frac{1}{100}$.) If we used $h = \frac{1}{300}$, we should get an error of roughly .0000028 (another two decimal places). For a picture, I used the following command in Mathematica:

```
Show[
 Plot[3t-3 4 Exp[-t], t, 0, 1, Axes -¿ True, AxesLabel -¿ "t", "x", +
 PlotStyle -> {Blue, Thickness[.01]}, PlotRange -> {0, 1.6} ],
 ListPlot[ ImprovedEulerMethod[f1, 0, 1, 1, 1/3] // N, PlotStyle -> Black],
 ListPlot[ ImprovedEulerMethod[f1, 0, 1, 1, 1/30] // N, PlotStyle -> Red]
 ]
```

   Here, `f1` was defined by `f1[t_, x_] = -x  3 t+`.



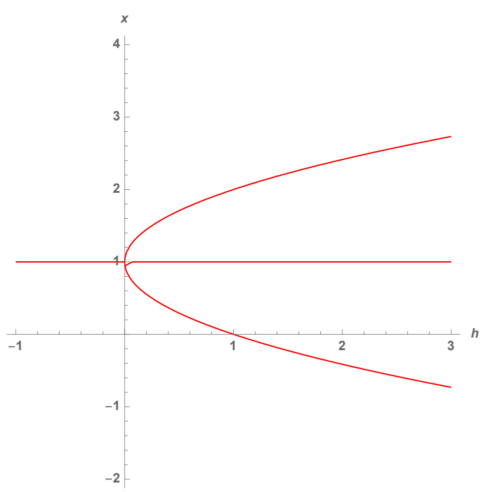
2. (5 points) Suppose the graph of $f(u)$ is given below. Draw the phase line for $u' = f(u)$, and determine the stability of the equilibria solutions.



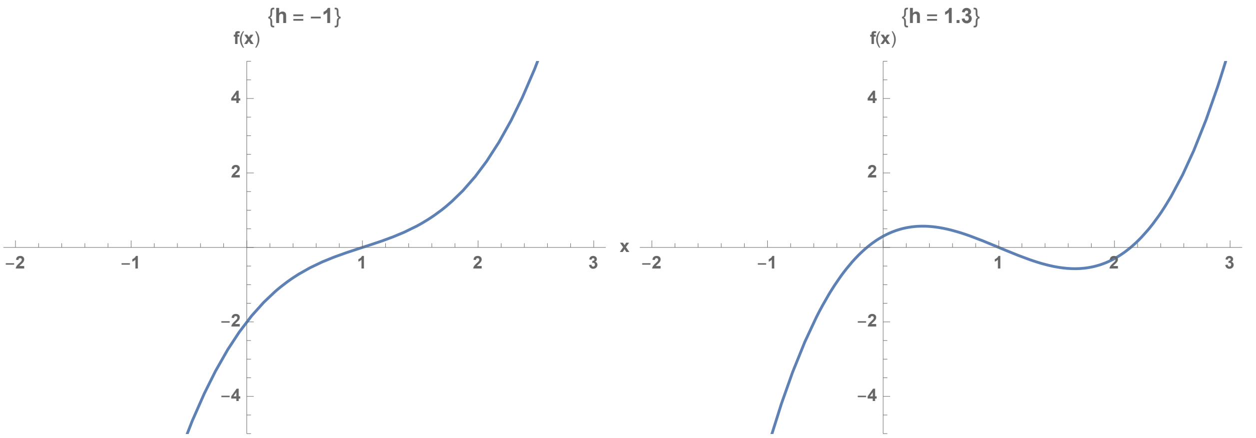
There should be FOUR equilibria: three negative ones, and a single positive one. In order from the most negative to the positive one, they should be stable, unstable, stable, and finally unstable.

3. (5 points) Construct a bifurcation diagram for $x' = (x-1)^3 - h(x-1)$, where h is the bifurcation parameter.

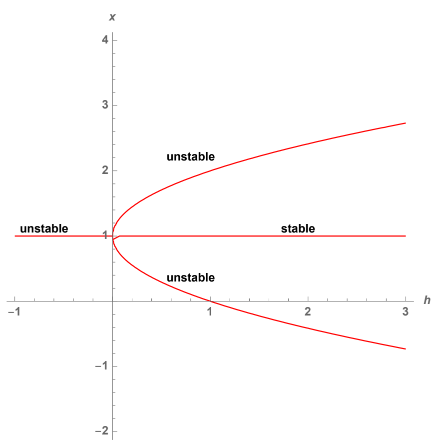
   We first plot all the points where $(x-1)^3 - h(x-1) = 0$. A little algebra gives us $(x-1)\big((x-1)^2 - h\big) = 0$. Notice that this tells us that $x = 1$ is always an equilibrium! In addition, we need to draw the curve $h = (x-1)^2$. Thus, we get the following picture:



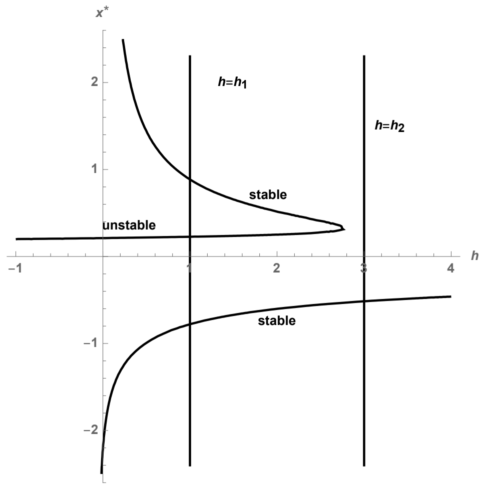
   Next, we pick a few representative values of h, and plot $x' = (x-1)^3 - h(x-1)$ for those values of h. In this situation, we need only plot what happens when $h < 0$ and $h > 0$, since whenever $h < 0$, there is a single equilibrium at $x = 1$, and when $h > 0$, there are always three equilibria: one smaller than 1, 1, and one larger 1. Notice that x' equals a cubic with a leading positive coefficient. That means regardless of the value of h, the graph of $x' = (x-1)^3 - h(x-1)$ will be a cubic that goes to $-\infty$ as $x \to -\infty$, and goes to $+\infty$ as $x \to +\infty$. When $h < 0$, we get a plot like on the left below, while when $h > 0$, we get a plot like the one on the right below (since we know that there are at least three zeros).



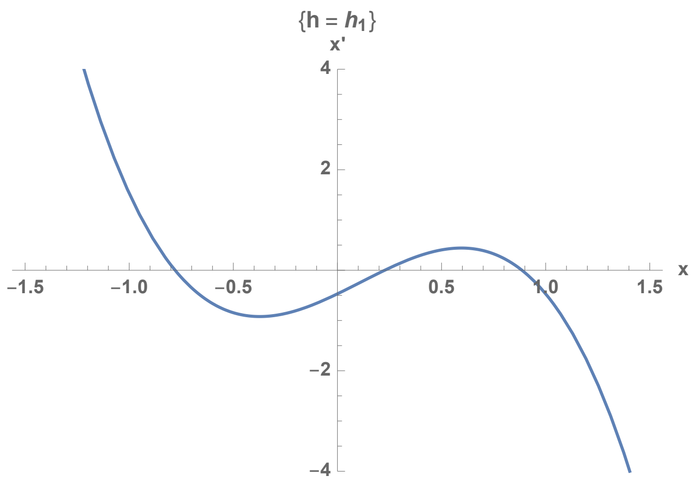
Therefore, when $h < 0$, the single equilibrium at 1 is unstable. When $h > 0$, the equilibria are (from smallest to largest) unstable, stable, and unstable. Note that the stability of 1 changes at $h = 0$. Thus, we have the following bifurcation diagram:



4. (5 points) The plot below is a bifurcation diagram for $x' = f(x; h)$. Draw appropriately labelled possible graphs for $x' = f(x; h_1)$ and for $x' = f(x; h_2)$, where h_1 and h_2 are the indicated values of h.



When $h = h_1$, there must be three equilibria, one which is negative, and two which are positive. The negative equilibrium must stable, and the larger positive equilibrium must also be stable, while the smaller positive equilibrium must be unstable.



When $h = h_2$, there will be a single negative equilibrium, which must stable: