Math 376 Homework Set #4 Solutions

1. (5 points) A container with capacity 20 liters initially holds 10 liters of water, with a concentration of 2 grams per liter. Suppose water with a concentration of $5te^{-2t}$ grams per liter flows in at 2 liters per minute, while the well-mixed mixture flows out at 1 liter per minute. What is the concentration inside the container when it overflows?

First, we determine the volume of water inside the tank at time t. We have $\frac{dV}{dt} = 2-1 = 1$, and therefore (by integrating) we have $V(t) = t + C$. Since $V(0) = 10$, we see that $V(t) = t + 10$. Next, we find an initial value problem for the mass in the container at time t. Conservation of mass tells us $\frac{dm}{dt} =$ (rate mass enters) $-$ (rate mass leaves). Therefore, we will have $\frac{dm}{dt} = (2)(5te^{-2t}) - \frac{m}{t+10}$, and $m(0) = 10 \cdot 2 = 20$. We solve the initial value problem. Note that the differential equation is a first order linear equation. We rewrite: $\frac{dm}{dt} + \frac{1}{t+10}m = 10te^{-2t}$. Next, we calculate the integrating factor $\mu(t)$: $\mu(t) = e^{\int \frac{1}{t+10}\,dt} = e^{\ln|t+10|} = |t+10| = t+10$, since $t \geq 0$. Multiplying the equation by $t+10$, we have $(t+10)\frac{dm}{dt} + m = (10t^2 + 100t)e^{-2t}$. Integrating both sides, we will have $(t+10)m = \displaystyle\int (100t + 10t^2)e^{-2t}\,dt$. We now compute $\displaystyle\int (100t + 10t^2)e^{-2t}\,dt$, using tabular integration.

u	$(100t + 10t^2)$	$(100 + 20t)$	20	0
dv	e^{-2t}	$-\frac{1}{2}e^{-2t}$	$\frac{1}{4}e^{-2t}$	$-\frac{1}{8}e^{-2t}$

Therefore, we have

$$\begin{aligned}(t+10)m &= -\frac{1}{2}e^{-2t}(100t + 10t^2) - \frac{1}{4}e^{-2t}(100+20t) - \frac{1}{8}e^{-2t}(20) + C \\ &= e^{-2t}\left(-50t - 5t^2 - 25 - 5t - \frac{5}{2}\right) + C \\ &= e^{-2t}\left(-5t^2 - 55t - \frac{55}{2}\right) + C,\end{aligned}$$

and so $m(t) = \dfrac{e^{-2t}}{t+10}\left(-5t^2 - 55t - \dfrac{55}{2}\right) + \dfrac{C}{t+10}$. Next, we need $m(0) = 20$, so we need $20 = \frac{1}{10}\left(-\frac{55}{2}\right) + \frac{C}{10}$. We now solve for C: $200 = -\frac{55}{2} + C$, so $C = 200 + \frac{55}{2} = \frac{455}{2}$. Therefore, we have,

$$m(t) = \frac{e^{-2t}}{t+10}\left(-5t^2 - 55t - \frac{55}{2}\right) + \frac{455}{2(t+10)}.$$

Now, **concentration** at time t is given by $\frac{m(t)}{V(t)}$, and the tank will overflow when $V(t) = 20$, i.e. when $t = 10$. Thus, the concentration when the tank overflows is given by:

$$\frac{m(10)}{V(10)} = \frac{e^{-20}}{20^2}\left(-500 - 550 - \frac{55}{2}\right) + \frac{455}{2(20^2)} \approx .56875.$$

2. (5 points) Find the solution of $x' = -\frac{4}{t}x + \cos(t^5)$, $x(1) = 0$.

This is a linear first order equation. We rewrite it as $x' + \frac{4}{t}x = \cos(t^5)$. Next, we calculate the integrating factor. We will have $\mu(t) = e^{\int \frac{4}{t}\, dt} = e^{4 \ln|t|} = e^{\ln|t^4|} = |t^4| = t^4$. Multiplying the equation by $\mu(t)$, we get $t^4 x' + 4t^3 x = t^4 \cos(t^5)$. We rewrite this as $\left(t^4 x\right)' = t^4 \cos(t^5)$. Integrating both sides, we get $t^4 x = \displaystyle\int t^4 \cos(t^5)\, dt$. We calculate the integral using substitution: let $u = t^5$, so $\frac{1}{5}\, du = t^4\, dt$. Then

$$\int t^4 \cos(t^5)\, dt = \int \frac{1}{5} \cos(u)\, du = \frac{1}{5}\sin(u) + C = \frac{1}{5}\sin(t^5) + C.$$

Therefore, we will have

$$t^4 x = \frac{1}{5}\sin(t^5) + C, \text{ and so } x(t) = \frac{\sin(t^5)}{5t^4} + \frac{C}{t^4}.$$

Finally, we solve for C: $x(1) = 0$ mean $0 = \frac{\sin(1)}{5} + C$ so $C = -\frac{\sin(1)}{5}$. Therefore, the solution of the initial value problem is $x(t) = \frac{\sin(t^5)}{5t^4} - \frac{\sin(1)}{5t^4}$.

3. Consider the initial value problem $x' = \dfrac{x^{\frac{2}{3}}}{\sqrt{t+1}} + t$, $x(t_0) = x_0$. For which values of t_0 and x_0 can you guarantee that there is a unique solution?

Here, we have $x' = f(t,x)$, where $f(t,x) = \frac{x^{\frac{2}{3}}}{\sqrt{t+1}} + t$. Since $f(t,x)$ is continuous everywhere it is defined a unique solution will exists so long as $t + 1 > 0$, i.e. $t > -1$. To guarantee that the solution is unique, we also need to know if $\frac{\partial f}{\partial x}$ is continuous. In this situation, we will have $\frac{\partial f}{\partial x} = \frac{2}{3}\frac{1}{x^{\frac{1}{3}}\sqrt{t+1}}$, which will be continuous as long as $t > -1$ and $x \neq 0$. Therefore, we can guarantee a unique solution for $t > -1$ and $x \neq 0$.

4. (5 points) For the IVP $x' = -x + 3t$, $x(0) = 1$, use Euler's Method with $h = \frac{1}{3}$ (by hand) and $h = \frac{1}{30}$ (with *Mathematica*) to approximate $x(1)$. Plot both your approximations as well as the actual solution and calculate the error between your approximations and the actual solution at $t = 1$.

Notice that we have $f(t,x) = -x + 3t$. Therefore, for $h = \frac{1}{3}$, you should have:

t	x
$t_0 = 0$	$x_0 = 1$
$t_1 = \frac{1}{3}$	$x_1 = x_0 + hf(t_0, x_0) = 1 + \frac{1}{3}(-1 + 3 \cdot 0) = \frac{2}{3}$
$t_2 = \frac{2}{3}$	$x_2 = x_1 + hf(t_1, x_1) = \frac{2}{3} + \frac{1}{3}\left(-\frac{2}{3} + 3 \cdot \frac{1}{3}\right) = \frac{2}{3} + \frac{1}{3}\left(\frac{1}{3}\right) = \frac{6}{9} + \frac{1}{9} = \frac{7}{9}$
$t_3 = 1$	$x_3 = x_2 + hf(t_2, x_2) = \frac{7}{9} + \frac{1}{3}\left(-\frac{7}{9} + 3 \cdot \frac{2}{3}\right) = \frac{7}{9} + \frac{1}{3}\left(-\frac{7}{9} + \frac{18}{9}\right) = \frac{21}{27} + \frac{11}{27} = \frac{32}{27}$.

This means that the Euler Method with $h = \frac{1}{3}$ implies that $x(1) \approx \frac{32}{27}$. Using Mathematica, the Euler Method with $h = \frac{1}{30}$ should imply that $x(1) \approx 1.44665$.

Next, a straightforward calculation (which you should be able to do!) shows that the solution of the IVP is $x(t) = 3t - 3 + 4e^{-t}$. Thus, the actual value of the solution at $t = 1$ is $x(1) = \frac{4}{e} \approx 1.47152$. We have the following error:

h	error
$\frac{1}{3}$	$\left\|\frac{32}{27} - \frac{4}{e}\right\| \approx .286333\ldots$
$\frac{1}{30}$	$\left\|1.44665 - \frac{4}{e}\right\| \approx .02487$
$\frac{1}{300}$	$\left\|1.46906 - \frac{4}{e}\right\| \approx .0024578$