Math 376 Homework Set #3 Solutions

1. (5 points) Find a one parameter family of solutions of $\frac{dx}{dt} = \left(\frac{x^2-6x+8}{2}\right) e^{-t}$.

   Notice that this is a separable equation! Rewriting, we have $\frac{2}{x^2-6x+8}\ dx = e^{-t}\ dt$.

   We next integrate both sides. Note: you calculate the integral of the right hand side on homework set #1! We have $\displaystyle\int \frac{2}{x^2-6x+8}\ dx = \ln\left|\frac{x-4}{x-2}\right| + C$. Therefore, after integrating both sides of $\frac{2}{x^2-6x+8}\ dx = e^{-t}\ dt$ and collecting constants, we will have $\ln\left|\frac{x-4}{x-2}\right| = -e^{-t} + C$.

   We now solve for x. Exponentiating both sides, we see $\left|\frac{x-4}{x-2}\right| = e^C e^{-e^{-t}}$ and so $\frac{x-4}{x-2} = \pm e^C e^{-e{-t}}$. Relabeling $\pm e^C$ as C, we have $\frac{x-4}{x-2} = Ce^{-e^{-t}}$. This is an equation of the form $\frac{x-4}{x-2} = \square$. We will have $x - 4 = x\square - 2\square$. and so $x - x\square = 4 - 2\square$. This means $x(1-\square) = 4 - 2\square$ and therefore $x(t) = \dfrac{4-2\square}{1-\square} = \dfrac{4-2Ce^{e^{-t}}}{1-Ce^{e^{-t}}}$.

2. (5 points) A can of your favorite beverage measures 68 degrees. You put it into a refrigerator that is 36 degrees. When you check 10 minutes later, the can is 60 degrees. How long until you can enjoy your favorite beverage at its best tasting temperature of 45 degrees? (And people say differential equations isn't useful!)

   Let $T(t)$ be the temperature of the can at time t. By Newton's Law of Cooling, we know that $\frac{dT}{dt} = k(36-T)$. We also know that $T(0) = 68$ and $T(10) = 60$. We need to find t such that $T(t) = 45$.

   We first solve $\frac{dT}{dt} = k(36-T)$. Notice that this is a separable equation, which we may rewrite at $\frac{1}{36-T}\ dT = k\ dt$. Integrating both sides and collecting constants, we will have $-\ln|36-T| = kt + C$. Solving for T, we see $\ln|36-T| = -kt + C$ and so $|36-T| = e^C e^{-kt}$, which means that $36 - T = Ce^{-kt}$ and to $T(t) = 36 - Ce^{-kt}$.

   We now determine C and k from the given information. From $T(0) = 68$, we se that $68 = 36 - C$, which means that $C = 36 - 68 = -32$. Thus, $T(t) = 36 + 32e^{kt}$. Next, since $T(10) = 60$, we see that $60 = 36 + 32e^{-10k}$. We now solve for k. $32e^{-10k} = 24$, and so $-10k = \ln\left(\frac{24}{32}\right)$, which means that $k = -\frac{1}{10}\ln\left(\frac{3}{4}\right) \approx .02877$

   Finally, we solve $T(t) = 45$ for t. We have $45 = 36 + 32e^{-kt}$, and we now solve for t. $9 = 32e^{-kt}$, so $\frac{9}{32} = e^{-kt}$, and thus $\ln\left(\frac{9}{32}\right) = -kt$, which means that $t = -\frac{1}{k}\ln\left(\frac{9}{32}\right) = \frac{10}{\ln\left(\frac{3}{4}\right)}\ln\left(\frac{9}{32}\right) \approx 44.1$ minutes.

3. (5 points) Find the solution of the IVP $x' - tx = t^3$, $x(1) = 2$.

   This is a first order, linear equation, with integrating factor $\mu(t) = e^{\int -t\, dt} = e^{-\frac{t^2}{2}}$. Multiplying both sides of the equation by $\mu(t)$, we get

$$e^{-\frac{t^2}{2}} x' - te^{-\frac{t^2}{2}} x = t^3 e^{-\frac{t^2}{2}},$$

   which we rewrite as

$$\left(e^{-\frac{t^2}{2}} x\right)' = t^3 e^{-\frac{t^2}{2}}$$

   We now integrate both sides. The integral of the left side is $e^{-\frac{t^2}{2}} x + C$. To calculate the integral of the right side, we use integration by parts with $u = t^2$ and $dv = te^{-\frac{t^2}{2}}\, dt$. Then $du = 2t\, dt$ and $v = -e^{-\frac{t^2}{2}}$ (to integrate $te^{-\frac{t^2}{2}}\, dt$, use substitution):

$$\int t^3 e^{-\frac{t^2}{2}}\, dt = -t^2 e^{-\frac{t^2}{2}} - \int 2t \left(-e^{-\frac{t^2}{2}}\right) dt = -t^2 e^{-\frac{t^2}{2}} + \int 2te^{-\frac{t^2}{2}}\, dt = -t^2 e^{-\frac{t^2}{2}} - 2e^{-\frac{t^2}{2}} + C$$

   Thus, after integrating both sides and collecting constants, we will have $e^{-\frac{t^2}{2}} x = -t^2 e^{-\frac{t^2}{2}} - 2e^{-\frac{t^2}{2}} + C$. Multiplying both sides by $e^{\frac{t^2}{2}}$, we see that

$$x(t) = -t^2 - 2 + Ce^{\frac{t^2}{2}}$$

   Finally, we find C so that $x(1) = 2$. We need $2 = -3 + Ce^{\frac{1}{2}}$ and so $5 = Ce^{\frac{1}{2}}$ and $C = 5e^{-\frac{1}{2}}$. Therefore, the solution of the initial value problem is

$$x(t) = -t^2 - 2 + 5e^{-\frac{1}{2}} e^{\frac{t^2}{2}} = -t^2 - 2 + 5e^{\frac{t^2 - 1}{2}}.$$

4. (5 points) Suppose the slope field given below is for the equation $u' = f(t, u)$. Draw a possible solution for the IVP $u' = f(t, u), u(0) = 1$.