Math 376 Homework Set #2 Solutions

1. (5 points) Is $x(t) = t\cos 4t$ a solution of $x'' + 16x = -8\sin 4t$? Why or why not?

   If $x(t) = t\cos 4t$, then $x'(t) = \cos 4t - 4t\sin 4t$ and $x''(t) = -4\sin 4t - 4\sin 4t + 16t\cos 4t$. Therefore, we will have

   $$x'' + 16x = -8\sin 4t + 16t\cos 4t + 16t\cos 4t = -8\sin 4t.$$

   Therefore, $x(t) = t\cos 4t$ does solve the equation $x'' + 16x = -8\sin 4t$.

2. (5 points) Consider the differential equation $u' = (u^2 - 4)(u + 1)$. Find all the constant solutions and draw them on an appropriately labeled set of axes. How do non-constant solutions behave -where are they decreasing or increasing? Why?

   The constant solutions occur when $u' = 0$, i.e. $(u^2 - 4)(u + 1) = 0$. Thus, we must have $u = \pm 2$, or $u = -1$. Therefore, there are three constant solutions: $u = -2, -1, 2$.

   The behavior of non-constant solutions depends on where they are: are they less than -2, between -2 and -1, between -1 and 2, or larger than 2.

   (a) Non-constant solutions less than -2: if $u < -2$, then $(u^2 - 4) > 0$ and $u + 1 < 0$. Thus, if $u < -2$, then $u' = (u^2 - 4)(u + 1) < 0$, which means that non-constant solutions less than -2 will decrease (since their derivative will be negative).

   (b) Non-constant solutions between -2 and -1: if $-2 < u < -1$, then $(u^2 - 4) < 0$ and $u + 1 < 0$. Thus, if $-2 < u < -1$, then $u' = (u^2 - 4)(u + 1) > 0$ and so non-constant solutions between -2 and -1 will increase (since their derivative will be positive).

   (c) Non-constant solutions between -1 and 2: if $-1 < u < 2$, then $(u^2 - 4) < 0$ and $u + 1 > 0$. Thus, if $-1 < u < 2$, then $u' = (u^2 - 4)(u + 1) < 0$ and so non-constant solutions between -1 and 2 will decrease (since their derivative will negative).

   (d) Non-constant solutions larger than 2: if $u > 2$, then $(u^2 - 4) > 0$ and $u + 1 > 0$. Thus, if $u > 2$, then $u' = (u^2 - 4)(u + 1) > 0$ and so non-constant solutions larger than 2 will increase (since their derivative will be positive).

3. (5 points) From class, we know a simple pendulum of length ℓ and mass m satisfies $\frac{m\ell^2}{2}\left(\frac{d\theta}{dt}\right)^2 + mg\ell\,(1 - \cos\theta) = \text{constant}$. Show that any non-constant motion $\theta(t)$ must satisfy $\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0$. (Hint: use implicit differentiation with respect to t.)

We differentiate both sides of $\frac{m\ell^2}{2}\left(\frac{d\theta}{dt}\right)^2 + mg\ell\,(1-\cos\theta) = \text{constant}$ with respect to t, noting that θ is really a function of t. This means that we must use the chain rule to calculate the derivative of $\left(\frac{d\theta}{dt}\right)^2$, and $1-\cos\theta$. We will have $\frac{d}{dt}\left(\left(\frac{d\theta}{dt}\right)^2\right) = 2\frac{d\theta}{dt}\frac{d}{dt}\left(\frac{d\theta}{dt}\right) = 2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2}$, and $\frac{d}{dt}\Big(1-\cos\theta\Big) = \sin(\theta)\frac{d\theta}{dt}$. Thus:

$$\frac{d}{dt}\left(\frac{m\ell^2}{2}\left(\frac{d\theta}{dt}\right)^2 + mg\ell\,(1-\cos\theta)\right) = \frac{d}{dt}\Big(\text{constant}\Big), \text{ and so}$$

$$\frac{m\ell^2}{2}\frac{d}{dt}\left(\left(\frac{d\theta}{dt}\right)^2\right) + mg\ell\frac{d}{dt}\Big(1-\cos\theta\Big) = 0, \text{ which in turn implies}$$

$$m\ell^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + mg\ell\sin\theta\frac{d\theta}{dt} = 0.$$

Thus, if $\frac{d\theta}{dt}$ is not identically zero (i.e. $\theta(t)$ is non-constant), dividing both sides by $m\ell^2\frac{d\theta}{dt}$ gives

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0.$$

4. (5 points) Find the general solution of $x' = \dfrac{t^3(1+x^3)}{x^2}$.

This is a first-order, separable equation. Therefore, we rewrite it as $\frac{x^2}{1+x^3}\ dx = t^3\ dt$. We next integrate both sides. On the left, we use substitution: let $u = 1+x^3$, so $du = 3x^2\ dx$. Therefore, will have

$$\int \frac{x^2}{1+x^3}\ dx = \int \frac{\frac{1}{3}}{u}\ du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|1+x^3| + C.$$

Thus, after integrating both sides, we will have $\frac{1}{3}\ln|1+x^3| = \frac{t^4}{4} + C$. We next solve for x:

Multiplying both sides by 3, we get $\ln|1+x^3| = \frac{3t^4}{4} + C$. Exponentiating both sides, $|1+x^3| = e^C e^{\frac{3t^4}{4}}$. Thus, we will have $1+x^3 = Ce^{\frac{3t^4}{4}}$, and so $x^3 = Ce^{\frac{3t^4}{4}} - 1$. Taking cube roots, we get

$$x(t) = \left(Ce^{\frac{3t^4}{4}} - 1\right)^{\frac{1}{3}}.$$