The Effects of Sudoku on Mathematical Problem Solving Ability in College Students

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Table of Contents

Approval .......................................................................................................................... 3
Acknowledgements .......................................................................................................... 4
Table of Contents .............................................................................................................. 5
CHAPTER 1 PURPOSE OF STUDY .............................................................................. 8
Purpose Statement ........................................................................................................... 8
  Problem Statement and Brief Introduction of the Problem .............................................. 8
  Background of the Problem .......................................................................................... 8
  Significance of the Study ............................................................................................. 10
  Research Questions ..................................................................................................... 10
CHAPTER 2 REVIEW OF THE LITERATURE ................................................................ 12
Overview of Problem Solving ........................................................................................ 12
  Polya, the Father of Modern Problem Solving Education .............................................. 12
  Preliminary Investigation by Researcher and Disclosure of Researcher Bias ................ 14
  Problem Solving Education: What Recent Research Suggests .................................... 16
Math Skills and Attitudes in Preservice Elementary Educators ...................................... 31
Journaling and Meta-cognition ...................................................................................... 32
Visuospatial Memory/Visual Abstract Processing .......................................................... 33
On Cognitive Benefits of Puzzle Solving ...................................................................... 36
Sudoku History .............................................................................................................. 41
Strategies for Solving Sudoku ...................................................................................... 44
Cognitive Skills Which May be Exercised by Sudoku .................................................... 46
Sudoku and Associative Memory .................................................................................... 46
Availability and Accessibility of Sudoku ....................................................................... 49
Summary and Conclusion .............................................................................................. 50
CHAPTER 3 METHOD ................................................................................................. 53
Participants ..................................................................................................................... 54
Apparatus ....................................................................................................................... 56
Procedure ....................................................................................................................... 56
Results, Email Interviews ................................................. 64

What grade did you get in your math class last quarter? Do you feel that Sudoku has helped in any way? If so, how?.............................................................. 64

Have you been practicing Sudoku? If so, how often? What levels can you do? ........ 64

Please read the next section, and tell me if you’ve noticed any improvement in yourself in these areas, or any other area you may have noticed. Sudoku has been said to enhance many cognitive skills. Among these skills are:................................................. 65

  Short term or working memory capacity ............................................. 65
  Pattern recognition ........................................................................... 65
  Tolerance for ambiguity and frustration ........................................... 66
  Attention span .................................................................................. 66
  Lateral thinking ................................................................................ 66
  Flexibility in problem solving............................................................. 67
  Other comments ............................................................................... 67

Discussion ........................................................................................ 68

CHAPTER 4 RESULTS ........................................................................ 59

Case 1: Linda, Chemistry Education .................................................. 59

Case 2: Lucy, English Education ......................................................... 60

Case 3: Erin, Biology .......................................................................... 62

CHAPTER 5 SUMMARY CONCLUSION AND RECOMMENDATIONS ..... 69

Summary and Conclusion ................................................................... 69

Recommendations .............................................................................. 69

REFERENCES .................................................................................. 71

APPENDIX A – Survey Instruments ..................................................... 79

APPENDIX B Euler, De Quadratis Magicis (On Magic Squares), annotated ........ 81
ABSTRACT

Current educational standards demand problem solving instruction. However, many preservice educators lack problem solving skills. An extensive literature review suggests that various methods, including combinations of direct instruction and discovery, can be effective in teaching problem solving. Puzzles and play are especially helpful. Sudoku, a popular logic puzzle, is offered to make problem solving accessible and enjoyable. Fourteen college students, including preservice educators, were introduced to Sudoku, yielding three case studies and six interviews. Participants reported improvements in memory, attention span, lateral thinking, tolerance for frustration, and math class grades. Results suggest that Sudoku is an appropriate addition or enrichment to a problem solving curriculum; further study is warranted.

Euler’s 1776 paper, De quadratis magicis, translated by J. Bell and annotated by the researcher for modern readability, is included as Appendix B.
CHAPTER 1 PURPOSE OF STUDY

Purpose Statement

The purpose of this study is to determine if regular practice of Sudoku improves mathematical problem solving ability in college students. In particular, the components of problem solving ability to be studied are:

1) Cognitive: visuospatial working memory, measured by Klein Progressive Matrices Test.

2) Affective: attitudes around problem solving, measured by Likert scale surveys, journal entries kept by participants and the researcher, and email interviews with participants.

3) Psychomotor: final grade in current mathematics class, compared with expected grade as expressed by participant at beginning of study.

Problem Statement and Brief Introduction of the Problem

Federal and state educational standards demand an increase in problem solving ability in today's students (U.S. Congress, 1994, OSPI 2007). Although many programs and curricula are in place to teach problem solving ability (Washington University, 2008), many preservice elementary educators still lack problem solving skills (Buffalo State University, 2006). Consequently, a challenge is to find ways to teach problem solving to preservice educators in a manner that they can pass on to their own students. Moreover, these practices should be portable, accessible, and engaging.

Background of the Problem

Recent standards based on National Council of Teachers of Mathematics (NCTM) recommendations and No Child Left Behind (NCLB) mandates place heavy emphasis on
problem solving ability and a great demand on our education systems to teach this ability (NCTM, 1989; U.S. Congress, 1994). However, studies of the transferability of problem solving skills have had mixed results (Anderson, Reder, & Simon, 2000).

In regard to cognitive aspects of problem solving: Sudoku has much to offer. It is based on a mathematical construct that Euler (1776) considered as a study on combinatorics. Research suggests that it may mitigate age related cognitive decline (Coyle, 2003; Verghese et al, 2003). Sudoku is currently used in educational settings because of its availability and popularity, and because many people believe that it supports or enhances cognitive function in young people as well (Edwards, 2006; McCormick, 2005, Naylor, 2006, Writer, 2007). Sudoku is said to exercise logical thinking skills (Naylor, 2006), lateral thinking skills (McCormack, 2005), associative memory (Hopfield, 1982, 2006; see also Gurr, 1987 and Skatssoon, 2006), and many other components of problem solving as described by Polya (Polya, 1945, Wilson Fernandez & Hadaway, 1993). Thus, it is reasonable to suppose, as Sudoku enthusiasts would say, "Sudoku makes people smarter." This, loosely stated, is the cognitive research hypothesis. For the purpose of study, the researcher focused on one aspect of "smartness": visuospatial working memory. This aspect is chosen because visuospatial working memory has the following characteristics:

1. It is connected with Polya's meta-strategies for problem solving, which often rely on pictures and diagrams to represent problems and solutions (Polya, 1947, Wilson et al, 1993).
2. It is a basis for many Sudoku strategies, particularly those strategies which a naive player is likely to devise for themselves (Aslaksen, 2008, Intelm, 2003, Stevens, 2007).

3. It is readily tested and is a component of many standardized aptitude and achievement tests (Kyttälä, M. and Lehto, J, 2008).

4. It is demanding on the mental processing of visual imagery, which corresponds to a strong area in the normal human brain (Schmolesky, 2006).

In regard to affective aspects of problem solving: Sudoku is very popular and widely available. It is accessible to children as young as 7 years old, (Naylor, 2006) and it is not dependant on any particular language. Thus, it may improve attitudes toward problem solving by demonstrating that problem solving can be accessible, engaging, and rewarding. Participants may verify this for themselves if their grades in their current mathematics classes exceed their own expectations based on previous experience.

Significance of the Study

Practice of Sudoku may enhance problem solving ability in college students by exercising visualization and improving attitude and performance. In particular, prospective educators may benefit from this study, as may their students.

Research Questions

At the outset, this study was planned as a quantitative study with research questions as listed below. However, due to low participant attendance of follow up meetings, results are presented as case studies. These case studies support the association
between Sudoku practice and improved cognitive fitness leading to improved problem solving ability. The original research questions were as follows:

College students who are trained in and practice Sudoku will score significantly higher on a test of visuospatial ability than they had scored prior to training and practice.

College students who are trained in and practice Sudoku will score significantly higher on a Likert scale survey of attitudes toward problem solving than they had scored prior to training and practice.

College students who are trained in and practice Sudoku will exceed their own expectations in regard to their final grades in the math class in which they are currently enrolled, with these expectations based on their previous experience and reported prior to training and practice.

Although only three of the original fourteen participants reported for re-testing on the test of visuospatial ability and Likert scale survey, the researcher was able to obtain information on their math class grades and their attitudes toward problem solving through interviews. Additional information was gathered in these interviews about improvements in memory, lateral thinking, attention span, tolerance for frustration, and other aspects of cognitive fitness leading to improved problem solving ability.
CHAPTER 2 REVIEW OF THE LITERATURE

Overview of Problem Solving

: An extensive review of the literature follows, in keeping with accepted practice in the mathematics community to make minimal assumptions and to prove, or cite proofs, whenever this is warranted.

Many programs and curricula have been and are being developed to teach problem solving. It is still a matter of lively debate whether problem solving is best taught through discovery, direct instruction, or a combination of these approaches. This review will discuss these different approaches and under what circumstances each is appropriate. Cognitive benefits of puzzle solving, in particular visuospatial memory and associative memory (the latter originally a construct of artificial intelligence), and some considerations of evolutionary biology will be discussed. Disclosure of researcher bias, based on her experiences, is also included in this section.

Teaching Problem Solving

*Polya, the Father of Modern Problem Solving Education*

Most current methods of teaching problem solving are based on the four step method that Polya (1945) explained in his classic *How to Solve It* (Wilson et al, 1993). The four steps of Polya’s method are as follows:

Understand the problem

Make a plan

Carry out the plan

Check your work (Polya, 1945)
One must read Polya's book to see the complexity and richness of these steps. Understanding the problem means considering all available data, whether within the problem or nearby, the relations among them, and identifying what is known and what is needed for a solution. Making a plan means considering similar or simpler problems, searching through one's repertoire of methods for solving these problems, and considering how these methods may be applied, as is or as modified. Carrying out the plan requires perseverance, which in turn requires hope (a knowledge or belief that the problem can be solved, whether because similar problems have been solved, or simply out of naivety or trust in one's instructors) and some degree of familiarity (again, knowledge of similar problems or procedures). Finally, checking the work means checking to see if the solution is reasonable, if it fits all of the data and makes sense. To check the work also means to consider if this problem could have been solved some other way, and if its solution can be applied to other situations or problems (Polya, 1945).

Polya also stressed the importance of modeling, imitation, and having a rich repertoire of data and strategies at one's disposal. As he wrote on page 4: "Solving problems is a practical skill like, let us say, swimming. We acquire any practical skill by imitation and practice." (Polya, 1945). Furthermore Polya wrote:

"We know, of course, that it is hard to have a good idea if we have little knowledge of the subject, and impossible to have it if we have no knowledge. Good ideas are based on past experience and formerly acquired knowledge. Mere remembering is not enough for a good idea, but we cannot have any good idea without recollecting some pertinent facts; materials alone are not enough for
constructing a house but we cannot construct a house without collecting the
necessary materials. The materials necessary for solving a mathematical problem
are certain relevant items of our formerly acquired mathematical knowledge, as
formerly solved problems, or formerly proved theorems. Thus, it is often
appropriate to start the work with the question: *Do you know a related problem?*
(Polya, 1945, p.8, italics in original text)

In 1945, Polya wrote to an audience who would have been raised under traditional
content-rich methods of direct instruction, and whose difficulty would have been
considering "too many" known problems. That is to say, students of that era would have
memorized processes for solving various types of problems encountered in their science
and mathematics classes, without necessarily having an understanding of how to solve
new problems or how these memorized processes work. Polya noted this, and suggested
narrowing down the field: "Look at the unknown! And try to think of a familiar problem
having the same or similar unknown." (Polya, 1945, p. 9). Modern audiences may have a
different problem, in that they may know *too few* problems and their solutions (Anderson
et al, 2000). That is, modern students are often expected to explore and innovate with
only minimal grounding in established facts and procedures. However, the advice "Look
to the unknown!" is still worthwhile.

*Preliminary Investigation by Researcher and Disclosure of Researcher Bias*

As a preliminary investigation, the researcher has conducted informal interviews
with friends and neighbors. She has found that many non-Sudoku players are intimidated
by the grids and numbers. Moreover, many of the proficient Sudoku players she knows
did not "figure it out themselves" from the novice stage. Instead, they were introduced to
the game by another player, who provided an easy puzzle or two, and perhaps offered a
few starting moves. The game may be found in many newspapers and other public
venues and the caption usually reads "this game uses no math." Yet, the response of one
intelligent college educated person (with a Masters of Fine Arts degree and a successful
career as an advertising designer) was telling and typical: "All those numbers! Do I have
to add them up or something? I never was any good at math!" The researcher responded,
"See the caption? This game uses no math." The person retorted, "Why would I want to
go as far as reading the caption? All those numbers scare me!" On a wager, the person
allowed the researcher to show her a variant of the game using a smaller grid with letters
instead of numbers, and a simple start-up strategy. Within minutes, the person took
easily to the game, and soon discovered strategies of her own. (The researcher lost the
bet, and treated the person to a cup of cappuccino.)

Another source of researcher bias is her experience as a high school student in an
experimental curriculum where self directed self paced discovery learning was required.
In that situation, the more advanced students (such as the researcher) were forced into the
role of peer tutors. This experience as a high school student, and the researcher’s
subsequent experience as a professional tutor, has led her to believe that people do not all
develop Piaget’s Stage Four thinking at the same age, and that many people never
develop into this stage at all. (Piaget’s Stage Four thinking is also known as Formal
Operations, and may be regarded as the ability to think abstractly.) The researcher
believes that those who have not developed into this stage need specific guidance and
external motivation when learning problem solving. See also Vygotsky (1978).

These beliefs about the non-universality of Stage Four thinking and the need for explicit guidance and external motivation at the threshold of this stage are borne out by Genevese (2003). See also Huit & Lutz (2003) and Huit & Hummel (2003). Using these beliefs as a working model for providing individualized instruction, she has experienced many successes with students whose own previous experience in learning mathematics had been very discouraging.

The researcher’s preliminary investigation and life experiences have given her an experiential sense as to how to make problem solving in general and Sudoku in particular accessible to novices. This generally includes simplification, reframing, and other scaffolding tailored to the individual. Cotton (2003) suggested that the method of teaching problem solving ability does not seem to matter so much as the fit of the chosen method to student background, teacher training and commitment, and community and administrator support. Thus, the researcher feels some direct instruction may be helpful, even necessary, in introducing Sudoku to beginners, and that this direct instruction will not undermine their problem solving ability, enjoyment of the game, or cognitive benefits that may be gained from playing.

*Problem Solving Education: What Recent Research Suggests*

Polya was a brilliant mathematician and educator, and his insights into the problem solving process were developed through his own experience in the early 20th century as a university professor in Europe (O'Connor & Robertson, 2002). Therefore,
more recent research is considered next, particularly research on younger students and students in the United States.

Delclos and Harrington (1991), citing Dreyfus and Dreyfus (1986) and Novick (1988) noted that, according to Novick’s scheme, many middle school children are at the stage of “advanced beginner” in problem solving. They looked to research literature to determine what type of instructional strategies best meet the needs of learners at this stage. Delclos and Harrington stated that "problem solving instruction, then, should include (a) multiple examples of the functionality of new facts and strategies within the given domain and (b) the opportunity to practice using those tools to solve actual problems" Also, there is a self-feedback component: "Students must develop the ability to monitor their performance and come to recognize the importance of evaluating progress toward their goal" (Delclos and Harrington, 1991).

Delclos and Harrington suggested that knowledge must be provided within the context of the learners’ experience, and that this information should accompany opportunity for practice. They proposed two models of providing this instruction: “(a) deal with impasses as they arrive, taking an interactive coaching approach (see Delclos and Kulewicz, 1986) or (b) deal with impasses pro actively, teaching specific strategies along with the factual information in the domain.”

A previous study with Kulewicz (Delclos and Kulewicz, 1986) worked with novice learners and the educational computer game "Rocky's Boots." This study proved that the reactive approach of option (a) is more effective than non-intervention. Without teacher assistance, all children reached a plateau in their performance, and they advanced
beyond this plateau only by teacher intervention. (See also Vygotsky, 1978.) The results of the 1986 study by Delclos and Kulewicz enabled the authors to improve the game's built-in tutorial, and develop the role of the human teacher in computer based instruction. In the 1991 study with Harrington, Delclos asked whether a proactive teacher approach (b) is also effective, and if the additional treatment of self-monitoring makes any difference in problem solving proficiency.

In the 1991 study, middle school aged children were again asked to solve the computer game, "Rocky's Boots." In pre-treatment, the children were all familiarized with the rules and structures of the game, so that they were advanced beginners rather than novices according to Novick's scheme. There were three groups. The C (control) group received no problem solving training. The PS group received proactive problem solving training (that is, they were offered instruction on game strategy in an organized manner as the experiment progressed). The MPS group received pro-active problem solving training; also they received a checklist for self-monitoring. The results showed that both PS and MPS groups did better than the C group. Moreover, the MPS group did better than the PS group, particularly on the more complex problems toward the end of the game. Thus, pro-active instruction is more effective than no instruction, and self-monitoring enhances the effect of pro-active instruction.

In “Teaching Thinking Skills,” a review of 56 documents, Cotton (1991) found that “nearly all of the thinking skills programs and practices investigated were found to make a positive difference in the achievement levels of participating students.” Some of the successful programs reviewed here addressed study skills, creative and critical
thinking skills, meta-cognition, and inquiry training. Techniques as widely varied as computer assisted instruction, redirection / probing / reinforcement, and increased teacher wait time are all associated with cognitive gains. There exist equally strong support for infused (that is, incorporated into the curriculum) and separate programs, and for programs using direct instruction or inferential learning (Cotton, 1991).

Some programs involve "controlled floundering," where students must feel their way along, although teachers remain with them and assist them to work through the steps of their tasks. Other programs (here, Cotton cited Freseman, 1990, Herrstein et al, 1986, Pearson, 1982, and Wong, 1985) also proved effective; these favor direct instruction of the target thinking processes, with teacher modeling and student practice. These programs seem to work better for students "...whose out of school lives have offered little exposure to higher-order thinking [and who] cannot be expected to develop these skills inferentially and must be taught them directly. . . “(Cotton, 1991). In regard to direct instruction versus inferred learning, Cotton concluded, "It would appear that either approach can be effective, and a blend of the two may be most effective."

Cotton concluded that it is important and worthwhile to teach thinking skills, and there are a wide variety of successful methods available. Often the success of these programs depends on other factors. Cotton wrote:

…administrative support and appropriate match between the students and the instructional approach selected. Effective methods may be high or low tech, may use direct or inferred instruction, and may be infused into the curriculum or separate programs. In common, they require a great deal of time; teacher,
administrator and community commitment; and a positive, simulating, and encouraging classroom climate. (Cotton, 1991)

The above study by Cotton suggests that many forms of problem solving instruction can enhance problem solving skills. The most important factor seems to be to find a good fit between the method, the students, the teachers, the administration, and the community. Thus, in choosing a method of teaching problem solving, it would be good to follow the first rule of writers: "consider your audience."

On the other hand, Jerome Bruner made an eloquent case for discovery learning, and John Dewey had much success with this method in his Laboratory Schools. Proponents suggest that this method is superior for retention and transfer of knowledge (Bruner, 1972). Discovery based learning for problem solving is best supported by evidence from science classes, particularly at the high school and college level. Problem Based Learning originated as a method for training medical students for making diagnoses. In this form, still in use in medical schools, it includes a flowchart model for an orderly heuristic process, and some direct instruction (Hmelo-Silver, 2004).

In “Improving Mathematics Teaching” Stigler and Hiebert (2004) reported on a component of the Trends in International Mathematics and Science Study (TIMMS). Videos of eighth grade mathematics classes from Australia, Czech Republic, Hong Kong, Japan, the Netherlands, Switzerland, and the United States were analyzed for organization of lessons, mathematical content, and the ways in which the class worked on the content. (The videos from Switzerland were dropped from the study because English translations of these videos were not available.) The authors found the lessons from the
Czech Republic, Hong Kong, Japan, and the Netherlands had a majority percentage focused on making connections rather than using procedures. “Japanese students, for example, spent an average of 15 minutes working on each mathematics problem during the lesson, in part because students were often asked to develop their own solution procedures for problems that they had not seen before” (Stigler & Hiebert, 2004).

Lessons from Australia and the United States placed a stronger emphasis on procedures; in the case of the United States, this was virtually to the exclusion of making connections. This recalls the Foss and Kleinsasser study (2001) where preservice educators said that they stressed higher level thought but closer examination did not bear this out. The authors made several recommendations:

1. “Focus on the details of teaching, not teachers.” Teaching should be considered as a cultural activity, and its conventions challenged and changed.

2. “Become aware of cultural routines.” Analysis of practice should be applied to highlight the cultural routines of teaching in the U.S. and in other TIMMS countries. This analysis can then be used to implement changes in U.S. classroom culture.

3. “Build a knowledge base for the teaching profession.” Teachers need “theories, empirical research, and alternative images of what implementation looks like. . . . Teachers need access to examples. . . they need to analyze what happens when they try something new in their own teaching…they need to record what they are learning and share that knowledge with their colleagues.” (Stigler & Hiebert, 2004).
The researcher observes that, with the possible exception of Australia, the other schools in the TIMMS study are far more stratified by ability than schools in the United States. For example, in Japan, students must take entrance exams at junior high school (seventh grade), high school (tenth grade) and college level. As a result of these exams they are assigned to different programs for vocational, technical, or college preparation (Maciamo, 2004). Similar systems exist in the Czech Republic, (VeronikaB, n.d.) Hong Kong, (ITS, 2005) The Netherlands, (Dutch Eurydice Unit, 2007) and Switzerland, (Ambühler, 2007). The Australian system is also stratified by ability, although the Australian system does not seem to be as dependant on entrance testing as the systems in the other five high performing countries in the TIMMS study (Commonwealth of Australia 2006).

<table>
<thead>
<tr>
<th>Age</th>
<th>Grade</th>
<th>Preoperational</th>
<th>Entry Concrete</th>
<th>Advanced Concrete</th>
<th>Entry Formal</th>
<th>Middle Formal</th>
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<tbody>
<tr>
<td>14</td>
<td>8-9</td>
<td>1</td>
<td>32</td>
<td>43</td>
<td>15</td>
<td>9</td>
</tr>
<tr>
<td>15</td>
<td>9-10</td>
<td>1</td>
<td>15</td>
<td>53</td>
<td>18</td>
<td>13</td>
</tr>
<tr>
<td>16</td>
<td>10-11</td>
<td>1</td>
<td>13</td>
<td>50</td>
<td>17</td>
<td>19</td>
</tr>
<tr>
<td>16-17</td>
<td>11-12</td>
<td>3</td>
<td>19</td>
<td>47</td>
<td>19</td>
<td>12</td>
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<tr>
<td>17-18</td>
<td>12</td>
<td>1</td>
<td>15</td>
<td>50</td>
<td>15</td>
<td>19</td>
</tr>
</tbody>
</table>

The TIMMS study was eighth grade math classes. The chart above shows an estimate from longitudinal studies by Lutz & Huiitt, (2004, p.4) to the effect that about 24% of US eighth grade students are in early or middle Stage Four development, while 75% remain in Concrete Operations. (The other 1% functions at the level of preoperational. It is also interesting to note in the chart above that 3% of the participants age 16 to 17 appeared to regress to preoperational thought. This may be connected with the return to egocentricity and other primitive cognition patterns that some adolescents experience at the onset of Formal Operations. At any rate, this regression appears to be temporary. By age 17 to 18 this proportion returns to 1% as seen in younger teenagers, and probably represents the small number of people in the general population who have significant cognitive impairments.) Huiitt, & Hummel (2003) placed the estimate of ninth grade students who attain Stage 4 thought somewhat lower, at around 20%, as the following figure shows:
Also, Huitt, & Hummel cited data from Kuhn, Langer, Kohlberg & Haan, (1977) suggesting that only about 30 to 35 percent of U.S. high school seniors attain Stage 4 cognition, and that proportions for other industrialized countries are similar.

As noted above, most of the other schools in the TIMMS study have some sort of mechanism for separating students by ability at around this time in their lives. The researcher’s own experience (see “Disclosure of Researcher Bias,” above) suggests that students who have not yet begun to develop Stage Four thinking have a different Zone of Proximal Development from those who have approached or attained this stage (see Vygotsky 1978). Those who have not begun to approach Stage Four may need more scaffolding, and a different type of scaffolding, in order to make connections. Non-Stage Four thinkers may also make a different type of connections from Stage Four thinkers.

The researcher conjectures that U.S. teachers are overly burdened by the varying demands of heterogeneous ability levels at this crucial time in young people’s cognitive development. This may be true especially if they are trying to maintain Piaget’s beliefs that development into Stage Four is biologically driven, and that all humans of normal intelligence begin to develop Stage Four thinking at around age 11 and complete this transition by age 15 (Lutz & Huitt 2004 p4). Thus, the researcher suggests that U.S. teachers may resort to the path of least resistance, and devolve all teaching into exposition of procedures. This may also account for the anxiety reported by preservice educators concerning the NCTM standards (Foss and Kleinsasser 2001).

Nevertheless, stratification by ability is not acceptable to contemporary U.S. values. The researcher is certainly not recommending a return to this practice which has
led to civil rights abuses in the past. She is simply suggesting that perhaps this reduction to procedure is part of the price we pay for an equitable education system.

In "Discovery Learning and Transfer of Problem-Solving Skills”, McDaniel and Schlager (1990) concluded that "…discovering a strategy provided benefits when a new strategy had to be generated to solve a transfer problem, but not when the learned strategy could be applied, albeit with new moves, to the transfer problem” (McDaniel and Schlager, 1990).

The McDaniel and Schlager study consisted of two experiments. In the first experiment, participants attempted to solve the "Hobbits and Orcs” problem. Some participants were taught a strategy for problems of this type, while others were given no training and challenged to discover the solution for themselves. Then, all participants attempted a disguised variation on this problem, "Wives and Jealous Husbands.” Those who had discovered a solution to the "Hobbits and Orcs” problem did not solve the "Wives and Jealous Husbands” problem more proficiently than those who had been taught a strategy. McDaniel and Schlager wrote "…when transfer required new move sequences to implement a general strategy learned previously, discovery did not enhance transfer of that strategy” (McDaniel and Schlager, 1990).

In the next experiment of the McDaniel and Schlager study, participants solved the water jar problem. One group of participants was taught a general strategy, involving addition, for solving this type of problem; the other group was asked to discover a solution. After both groups had found solutions, they were then presented with a related problem which could not be solved by the learned strategy; that is, it involved subtraction
and multiplication as well as addition. Both groups found a solution to this problem also, although those in the discovery group found a solution faster. There was a slight transient negative effect in those students who had been trained in a strategy. At first, they did slightly worse than the control group, perhaps because they wasted time and effort by using their learned strategy. As Lundin et al. (2004) suggested, perhaps the wrong kind of problem solving training "undermines problem solving." However, this effect was transient, for the first few trials. This lag may have occurred until participants realized that the strategy they knew would not work, so they would have to find a new one. In later trials, the trained participants caught up, and outperformed the control participants. Thus, "…having to discover a strategy while in training produced better transfer than being provided with a strategy while in training" (McDaniel and Schlager, 1990).

There are a few items worthy of note in the McDaniel and Schlager study. In both experiments, participants were college students enrolled in a psychology class. In the first experiment, they were offered extra credit in their course; in the second, they were required to participate in this study as part of their course credit. Prior to both experiments, participants were asked to complete a puzzle; those who could not complete it within a specified time were given their course credit and dropped from the study. Thus, participants were mature learners, highly motivated to complete their part in the trial, and had already shown some sort of cognitive fitness for solving similar problems. It is noteworthy that participants who discovered a solution to the problem were more able to transfer their knowledge; however, these caveats suggest that discovery based learning may not be appropriate for all learners, or in all situations.
In "An Inquiry Primer" by Colburn (2000), Inquiry Based Instruction is defined as "the creation of a classroom where students are engaged in essentially open-ended, student-centered, hands-on activities" (p. 42). Colburn discussed several levels of Inquiry-Based instruction, in which progressively less information is provided to the students, and progressively more knowledge is constructed by the students. These levels are structured inquiry, guided inquiry, open inquiry, and learning cycle. Colburn, a proponent of this method, cautions that "students initially resist . . . but after several weeks they grow to like it, or at least appreciate its value” (p. 42). Thus, he suggested a gradual transition to this style of instruction in the classroom science lab. This transition seems to involve providing students with progressively less information. For example, first get students accustomed to generating their own data tables; then, provide them with only some of the procedures; have them attempt the activity before the lecture, etc. Colburn wrote:

Most studies state that inquiry-based instruction is equal to or superior to other instructional modes and results in higher scores on content achievement tests. However, some of these studies focused on students who were studying concrete content, which is the strength of inquiry-based education. . . . Research seems to support the idea that students can discover concrete concepts that lend themselves to direct observation through inquiry based education. . . . It's up to you to find the right mix of inquiry and non-inquiry methods that engages your students in the learning of science. (Colburn, 2000 p. 44).
Again, there is the recurrent theme that discovery learning is effective in science classes, with well-prepared learners, with concrete content, and that a mixed approach may be best. Also, there is some time, perhaps several weeks, which must be invested in overcoming student resistance and getting them accustomed to inquiry based instruction. This time is well-spent in preparing students for future scientific inquiry, perhaps in later lab-based classrooms, or in life. However, this study does not prove that inquiry based learning is more effective for abstract content, or for global gains in cognitive aptitude. Neither does it suggest that traditional information-rich methods of instruction may undermine cognitive aptitude or problem solving ability.

In "Piaget, Pedagogy, and Evolutionary Psychology", Genovese (2003) discussed the difference between “biologically primary abilities” and “biologically secondary abilities,” from a viewpoint of evolutionary biology. He suggested that biologically primary abilities correspond to the Piaget stages of sensory motor and concrete operations, in that these processes have been part of human consciousness long enough that the human brain is biologically adapted to them. That is, children learn to walk, manipulate, recognize, count and place objects in order; speak, recognize people, etc, effortlessly and universally because these skills are grounded in our Pleistocene past, and the human brain has adapted structures for these functions. In fact, many of these skills are shared by other mammals as well, especially other primates. However, skills such as reading, writing, higher level computations, and the formal operational skills required in abstract reasoning and problem solving are culturally determined, and are fairly new to our species, from an evolutionary time perspective. These biologically secondary skills
have not existed long enough to become intrinsic to our nervous systems. Thus, they do not come as easily and naturally to us as walking and talking. Instead, formal training is necessary which includes a period of direct instruction by the elders to the younger generation. In most industrialized societies, this means formal education with its requisite efforts, repetition, and extrinsic motivation. This may explain why children learn to walk and talk easily and effortlessly, with intrinsic motivation and with little or no formal instruction. However, children must often be externally motivated and formally instructed before they can read, perform advanced mathematics or science, or acquire many of the other skills of a modern literate society.

Genovese suggested that modern reform models of education have made a “fundamental error” in thinking that, because some types of learning are spontaneous, all types of learning can and should be. He explained that the practical skills learned in early childhood are our common heritage from our Pleistocene past; thus, they are "biologically primary" and built into the normal human brain. This type of learning is spontaneous and occurs in the generally the same time frames and manners in diverse cultures. (For example, children normally learn to walk at during their first year of life, and to talk between ages two and four, all over the world.) However, the skills taught in school were unknown to our ancestors until fairly recently. There may be a biological imperative for a child to learn to walk, talk, recognize and manipulate objects. However, there is no biological imperative for a child to learn the geography or language of distant places, or algebra, or any of the other "biologically secondary" skills expected of a
modern educated person. These skills are acquired only by dint of practice, instruction, and external motivation (Genovese, 2003).

Nevertheless, these skills are acquired, and become part of our common human heritage. We can learn algebra, for example; some of us even develop internal motivation to do so. However, we have not been doing algebra long enough for this skill to become intrinsic in the brain in the same way as the skills associated with concrete and sensorimotor operations. This may be why all humans of normal intelligence attain sensorimotor and concrete operational skills at about the same age and without any special effort, but only a minority of adolescents or adults in industrialized societies attain formal operations. Moreover, those who do attain formal operational thought often need environmental and educational support in order to do so. So again the questions are raised, is it reasonable or legitimate to expect learners to discover for themselves abstract methods of problem solving, without external guidance or motivation? Is it reasonable to expect this within the limited period of time allowed for this study?

On the other hand, educational reformers such as Bruner, Rousseau, Piaget, and others, have noted the value of children's play in their learning. At the other end of the human lifespan, Verghese et al (2003) noted that cognitively demanding leisure activities for senior citizens are associated with reduced rates of dementia. Coyle (2003) suggested that they may do either because they build "cognitive reserve," or because such activities stimulate renewal and repair of brain cells in the hippocampus (an area of the brain which is active in memory formation and learning and which deteriorates in senile dementia). Thus, the researcher hopes that Sudoku, as a cognitively demanding leisure activity, may
stimulate the type of cognitive fitness necessary to develop the higher level problem solving abilities desired in modern education.

Math Skills and Attitudes in Preservice Elementary Educators

Why are we concerned about attitudes about mathematics in preservice elementary educators? In "Math Anxiety in Pre-service Elementary Schoolteachers" (Malinsky, Ross, Pannells, & McJunkin, 2006), the authors acknowledged that teacher attitudes toward mathematics can affect teacher effectiveness as well as student attitudes. Undergraduates in various majors were surveyed. The result showed that females were more likely to express math anxiety than males, and those in liberal arts programs more likely to express math anxiety than those in science programs. Moreover, math anxiety did not appear to decrease as preservice elementary educators gained some internship experience, as the authors had hypothesized that they would.

In "Title III: Strengthening Institutions‖ Buffalo State University administrators (2004) noted that liberal arts majors have lower SAT and ACT scores than do science majors. In particular, preservice elementary educators have low test scores and correspondingly high levels of mathematics anxiety.

In "Contrasting Research Perspectives" Foss and Kleinsasser (2001) discussed attitudes and beliefs about mathematics in preservice educators. The preservice educators in the study said that they incorporate problem solving and higher level thinking into their lessons, and their acceptable grades in their mathematics methods courses and practicum support this claim. However, in-depth interviews, classroom observations, and anonymous surveys revealed that their actual teaching practices incorporate very little if
any high level cognitive demands. This study also indicated that many preservice educators harbor serious anxiety about mathematics, higher level thinking, and the demands of the NCTM 1989 Principles and Standards. Moreover, many participants seemed to equate mathematics with numerical computation rather than with pattern recognition, problem solving, and higher level thought.

Journaling and Meta-cognition

In the study at hand, participants were asked to keep journals of their efforts and discoveries. Also, the researcher kept a journal. Since journaling is very demanding on one’s time and energy, it is necessary to justify this practice in this experiment.

Journaling can be a form of self-monitoring, and, in the Delclos and Harrington (1991) study, the self-monitoring group experienced greater gains than the other two groups. Additional support for the practice of journaling may be found in “Persona-based journaling: Striving for authenticity in representing the problem-solving process.” Liljedahl (2007), reported a study in which preservice elementary and secondary educators (E1, E2, S1, S2) were enrolled in four sections of two courses on problem solving skills. All participants worked in collaborative groups, and kept journals of their efforts. Those in the E1 and S1 groups were instructed to focus their journals on the problem solving process: "your efforts are to be recorded in journal format detailing your progress, successes, failures, frustrations, thoughts, and reflections regarding the problem” (Liljedahl, 2007). Those in groups E2 and S2 were made aware of the various persona or narrative voices they may assume: mathematician, narrator, and participant, and directed to “tell me the story of how they solved the problem” (2007). Both groups
experienced "transformations" in their attitudes and abilities regarding problem solving; however, the E2 and S2 groups' transformations were "more profound" (2007). This was also reflected in the E2 and S2 journals which were “much richer and more authentic (2007).” This study has implications in effective practices in journaling for learning and assessment, and in the meta-cognitive gains of self-monitoring as described above by Delclos et al.(1991).

In "Writing in Mathematics: Making it Work" Seto and Meel (2006) discussed the experiences of a licensed high school teacher who, as a graduate teaching assistant, assigned writing activities to her students in an introductory college mathematics class. These assignments consisted of mathematical biographies, minute papers, and journals. The teacher found that all forms of writing assignments produced gains in her decision making abilities and in students' awareness of their roles and expectations. This, in turn, improved teacher effectiveness and student learning, again reflecting the advantages of self-monitoring as described by Delclos et al. Seto and Meel concluded that, although writing assignments take extra time and effort, this need not be excessive, and the benefits are well worth the costs.

Visuospatial Memory/Visual Abstract Processing

For the purpose of data triangulation, the researcher is challenged to find an aspect of intelligence which is readily tested, which is demanded by mathematics, and which may be connected with those cognitive skills exercised by practice of Sudoku. Through conversations with Dr. S. Boersma and other Sudoku enthusiasts, the researcher concluded that non-verbal intelligence constructs such as Visuospatial Memory (as
measured by the Raven test, (originally developed by John C. Raven in 1936) or Visual Abstract Processing (as measured by the Weschler Adult Intelligence Scale, also known as WAIS) may be worth investigating.

In "Some factors underlying mathematical performance: The role of visuospatial working memory and non-verbal intelligence” Finnish researchers Kyttälä and Lehto, (2008) found that visuospatial memory is highly correlated with mathematical ability. Moreover, this skill can be improved by practice. The authors discussed why this correlation is reasonable, and verified it with the Raven test and the standardized mathematics test which has been developed to measure mathematical skills at various grade levels in Finland. Kyttälä and Lehto recommended that teachers encourage students to use computational and note-taking practices which exercise this ability, and that students incorporate such practices "both in and outside of school" (2008). While the authors do not specifically mention puzzles or games, their use may produce similar benefits. As Bruner noted, one of the purposes of play is to provide a safe and enjoyable setting for practice leading to proficiency (Bruner, 1972).

A test of nonverbal intelligence which is similar to the Raven test is readily available. Like the Raven test, this test uses "progressive matrices." These matrices are not matrices in the mathematical sense; instead, they are sequences of abstract drawings which become increasingly complex. The test uses no words except in the instructions, and places heavy demand on pattern recognition and visuospatial working memory. This test, to be referred to here as the Klein Progressive Matrices Test or the Klein Test, was developed by Sandor Klein and is published in his book *The Spirit of an Organization,*
not yet available in English. The author of this test is currently gathering data to validate it; therefore, the test is available online, free to the public, at www.egopont.com.

The Klein Progressive Matrices Test appears to test the same nonverbal intelligence as is measured by the Raven test. It consists of 18 sequences of abstract drawings similar to those used on the Raven test. It has advantages over the Raven test in that it is free, readily available, confidential, and can be self-administered in about 15 minutes, while the Raven test is copyright protected, expensive, must be professionally administered, and can take several hours.

The Klein Test is scaled to mimic the IQ tests which may be familiar to the public, such as the Stanford-Binet test or the WAIS test. Like the Stanford-Binet test and the WAIS test, the Klein Test has a mean of 100. This corresponds to a raw score of 14 correct out of 18 (average), with each additional correct answer over 14 correct corresponding to a 10 point increase in IQ. Thus, 16 out of 18 correct is assigned an IQ score of 120, 17/18 is assigned a score of 130 (“superior”), and 18/18 is assigned a score of 140. As Klein cautioned, this is “not a real IQ test” because it tests only one construct of nonverbal intelligence, and does so in a small number of trials. Klein also noted that his test addresses visual abstract processing skills, so a person with poor eyesight or poor visual memory would not expect to do well. Another consideration, not mentioned by Klein, is that the use of self-selected volunteers from the population of Internet users to set a mean score corresponding to an “average” IQ of 100 is likely to over-estimate the mean intelligence of the general population. That is, the average performance of Internet users who voluntarily take the test is likely to be higher than the average that would be
obtained if a truly random sample from the general population were used. In addition, the highest score one can obtain on the Klein test is 140, while it is possible to attain scores in excess of 200 on both the Stanford Binet and the WAIS intelligence tests. Therefore, the researcher did not record the IQ-like scores that the Klein Test generated, and she also cautioned participants not to give credence to these scores.

Participants took the Klein Test at the beginning of treatment as a baseline measure, then took it again after treatment to see if there had been any improvement. There was some possibility that they improved in their performance simply because of having seen the test before. To mitigate this, delay between test and retest was as least two months. As noted earlier, the IQ-like scores were disregarded. Only the raw scores were recorded, and only for the purpose of measuring gains in nonverbal intelligence which may be associated with Sudoku practice.

On Cognitive Benefits of Puzzle Solving

In "The Number's Up" published by the UK Government-backed Teachers Magazine, McCormick (2005) recommended that Sudoku and other puzzle games should be incorporated into the curriculum. In particular, McCormick said that Sudoku helps lateral thinking and “can stretch the most able mathematician” (2005). Subsequently, Teachers Magazine has included a Sudoku puzzle and solution in every issue, for use “inside or outside the numeracy hour” (2005).

Why should we expect puzzles to support learning? Considering puzzles as a form of play, the researcher looked to a seminal paper by Jerome Bruner (1972) “Nature and uses of immaturity.” Bruner noted that most other creatures live in an environment
provided by nature, use few or no artifacts, and carry most of the information needed for survival encoded in their genes. However, humans live in environments which are human-made, very rich in cultural artifacts, and which change and vary a great deal over time and geography. Our environments vary too much and change too quickly for the new behaviors to become encoded in our genes.

Because we lack genetic encoding of these behaviors, Bruner suggested that humans have a very long period of immaturity, a time characterized by play. In play, children enjoy a minimization of risks and consequences so that a safe learning environment is assured and a wide variety of activities can be attempted with impunity (p 693). In play, a child can repeat and vary the subroutines of a task until the task is mastered. During immaturity, children learn to negotiate the many innovations that humans have developed as part of our cultures, and reinforce these skills through play. Bruner suggested that one purpose of play is to rehearse problem-solving behavior in a safe and enjoyable setting, so that these skills will be well-honed and readily available should they be needed in a survival situation. Indeed, in traditional societies, there is often no formal schooling; instead, there is imitation of the elders and play. Gradually, as the child becomes an adult, the play-hunt or the play-escape become real-life struggles with survival at stake (Bruner, 1972).

In technological societies, the pace is accelerated, the content is increased, and formal schooling is necessary. Vast amounts of knowledge must be transmitted, and the young people must be motivated to practice and remain engaged. An "intermediate generation" of adolescents and young adults has emerged to engage in "deep play,” which
is characterized by high demands on technical expertise and greater risks to the participants. Some forms of "deep play" have some features in common with the initiation rituals of traditional societies, and they may serve some of the same psychological purposes. The urban skater, the sky diver, the body modification artist, all practice extreme forms of play in which one must know one's game or suffer the consequences. In these settings, the young "have more of a hand in teaching the younger" (p.695); thus, humans continue the practice of mastering the changing artificial environment, passing along knowledge through play (Bruner, 1972).

Although Sudoku is not physically demanding or dangerous, it may be considered a form of "deep play" in that it appeals to adolescents and adults, and it is highly engaging and demanding of technical expertise. An element of risk can be incorporated when Sudoku players compete at local, national, or worldwide championship levels. Sudoku has been implicated in at least one public scandal, when a drug trial was aborted because jurors were playing rather than paying attention to the hearings (Knox, 2008). While most forms of play are discarded in late childhood, deep play remains a feature of adolescence and adulthood, and a means by which humans can satisfy higher level needs for risk, competition, achievement, exploration, and continued learning (Bruner, 1972).

Why would humans be so interested in puzzles? What is the appeal? In The Puzzle Instinct, Danesi (2002) wrote that humans seem to have a natural desire for mystery, expressed in part in the worldwide appeal of puzzles. A puzzle creates suspense and arousal, which is resolved and rewarded when the puzzle is solved. This suspense
and resolution is similar to the feelings engendered by drama, mystery, and ritual. Danesi cited various forms of puzzles which are popular all over the world, such as labyrinths, riddles, number puzzles, and manipulatives. He credited many of these puzzles with innovations in science, mathematics, and the arts. He suggested that many cultures around the world enjoy puzzles, and that solving a puzzle is often a test of a hero’s worthiness. Danesi suggested that perhaps part of puzzles’ appeal is the sense of mastery that they give over an uncertain world. The successful puzzle solver may feel that, even though life is large and mysterious, the finite mysteries of the puzzle at hand are under control (Danesi, 2002).

For further support of the idea that solving puzzles may support cognitive fitness, the researcher looked to a study on senior citizens reported in 2003 in the New England Journal of Medicine. In “Leisure Activities and the Risk of Dementia in the Elderly,” Verghese et al. (2003) reported on a long term observational study that began in 1980. This study followed 469 participants who were between the ages of 75 and 85 years old at the beginning. In 1992, 73 surviving participants were still having study visits in a successor project, the Einstein Aging Study. Participants were monitored for regular participation in six cognitively demanding leisure activities, eight physically demanding leisure activities, and three physically demanding activities of daily living. Participants were tested by the Blessed Information Memory Concentration Test, the Weschler Adult Intelligence Scale (WAIS), the Fuld Object-Memory Evaluation, and the Zung Depression Scale to monitor ongoing cognitive fitness and the possible onset of senile dementia. "On average, subjects in whom dementia developed were older, had lower
levels of education, and had significantly lower scores on the cognitive activity scale, but not on the physical activity scale, than subjects in whom dementia did not develop” (Verghese et al, 2003, p 2511).

The question remains of “chicken or egg,” that is, whether senior citizens who were already in pre-clinical stages of dementia dropped leisure activities because of their declining cognitive fitness, or whether continued participation in such activities actively protected the elders from dementia. Verghese et al. attempted to address this difficulty by adjusting for education and for initial verbal IQ on the WAIS, and by eliminating from consideration those participants who were diagnosed with dementia within seven years of the start of the study. Even with these adjustments, the relation between cognitively demanding leisure activities and lowered rates of dementia remained significant.

In "Use it or Lose It--Do Effortful Mental Activities Protect against Dementia?” Coyle (2003) discussed the Verghese study and other studies on senile dementia. He mentioned the theory of "cognitive reserve," that is, the idea that "having greater intellectual resources (higher education, IQ, etc.) may buffer the underlying damage associated with the early stages of dementia, thereby delaying the onset of symptoms (Coyle, p 2489).” However, he called this view "static," and suggested that, to understand the impact of mental exercise, it is necessary to "give proper recognition to the remarkable use-dependent plasticity that characterizes the corticolimbic regions of the brain” (Coyle, p. 2490).

Conventional thought, based on the disappointing lack of recovery commonly seen in people with spinal and peripheral nerve damage, is that the
brain, also composed of nerve tissue, does not grow or repair itself in adulthood. However, with modern brain imaging technology such as MRI, we now know that this is not true (Bartzokis et al, 2003, Linden et al, 2007).

Effortful mental activity may not only strengthen existing synaptic connections and generate new ones; it may also stimulate neurogenesis, especially in the hippocampus. Thus, persistent engagement by the elderly in effortful mental activities may promote plastic changes in the brain that circumvent the pathology underlying the symptoms of dementia. (Coyle, p. 2490)

Thus, cognitively demanding leisure activity can promote neurogenesis even in senior citizens, conventionally thought to be in a state of irrevocable cognitive decline. It is reasonable to suppose that solving puzzles may also stimulate cognitive fitness for improved problem solving ability in young adult (college age) participants. This justifies the use of puzzles in education and in this study even though rigorous scientific proof of their effectiveness may still be lacking.

Sudoku History

Next, the history of Sudoku in relation to mathematics and popular culture is considered, to further support its use to assist development of mathematical problem solving ability. Sudoku is based on the earlier practice of "Magic Squares.” Euler (1776) presented a paper “On Magic Squares” in which he described a process for creating a “new kind of magic square.” Traditional magic squares, known to ancients, have the property that each row, column, and major diagonal sums to the same number. However,
repetition of numbers in a row or column is allowed. Euler’s “new type of magic square” had the constraint that in a \( x \) by \( x \) square, all numbers from 1 to \( x^2 \) occur in the grid, with each number occurring exactly once. There is a famous error at the end of Euler’s 1776 paper, which he corrected in 1782 in his follow up paper, "Investigations on a New Type of Magic Square. In the 1782 paper, Euler expanded on and proved the process that he had described in his earlier paper. He also computed the number of such squares for \( x \) taking on values from two to ten.

In his 1776 paper, Euler noted that for an \( x \) by \( x \) magic square, the sum of each row, column, and diagonal would be \( xx(1+xx)/2 \). (Euler uses the letter combination \( xx \) as we would write \( x^2 \).) Next, he stated what is now known as the remainder theorem, that given any two positive integers \( n \) and \( d \), these can be uniquely expressed as \( n = dq + r \), with \( d \) the divisor, \( q \) the integer quotient, and \( r \) an integer remainder between 0 and \( d \). Euler used Latin and Greek letters to keep track of quotients from 0 to \( x-1 \) and remainders from 1 to \( x \). (This is different from the usual statement of the remainder theorem, where remainders are from 0 to \( x-1 \); however, this statement is useful for the purpose of Euler’s construction.) Using this notation, Euler described how a magic square can be constructed using each number from one to \( x^2 \) in each row, column, and diagonal exactly once. It is interesting to note that Euler had become blind ten years earlier, and he did all of these calculations mentally.

Euler’s magic square is not the same as Sudoku because Euler used numbers from 1 to \( x^2 \) while a modern Sudoku uses the numbers from 1 to \( x \) (usually, \( x = 9 \)). However, a human interested in constructing a Sudoku puzzle by hand from a blank grid
could use a modification of Euler’s method, using only the Latin letters to represent the numbers 1 through x to place the “clues” or “seeds” along one row and one axis of symmetry. From these, one could create a Sudoku. (This is not how computers construct Sudoku puzzles; instead, computers do so by a modified brute force method which would not be feasible to humans.)

A copy of Euler’s 1776 paper, translated from Latin by Jordan Bell and annotated by the researcher for clarity to a modern audience, is included with this report as Appendix B. Writers for Nikoli Publishing use variants of Euler’s method to create “handmade Sudoku” puzzles. These writers take artistic pride in discovering and applying these variants, which are generally carefully guarded trade secrets. However, recently Nikoli (2005) produced a Sudoku book which includes a tutorial section revealing a few of these methods to the public.

Howard Garns, a 20th century retired architect and writer for Dell Puzzles, dropped the constraints on the diagonals, and replaced these with a constraint that the n by n grid is composed of smaller grids or "blocks." He also reduced the numbers used from \( n^2 \) to n, and required each number to appear exactly n times in the grid, once in each row, column, and block. Thus, he created the modern n by n Sudoku puzzle, first published by Dell in 1979 under the name “Number Place.” Number Place remained obscure until 1986, when Japanese publishing company Nikoli popularized the puzzle under the name “Sudoku” which means “single numbers.” Writers for Nikoli constructed puzzles by hand, a slow process which limited their availability. Sudoku did not become popular in the West until 2004, when retired judge Wayne Gould completed the first
commercially successful computer program to create Sudoku puzzles. He sold his puzzles to *The Times* in Great Britain, and now he is the main supplier of Sudoku puzzles worldwide. (Intelm, 2005).

Sudoku may thought of as a grid partitioned into sets of rows, columns, and blocks, and each partition contains exactly one occurrence of the numbers 1 through *n*. In the context of mathematics, the problem of solving a sudoku puzzle is the same as the vertex coloring problem in graph theory (Crook, 2009). Most modern Sudoku puzzles available in the mass media are composed of 81 elements, that is, nine rows, nine columns, and nine blocks; however, four by four, six by six, 12 by 12, and 16 by 16 puzzles are also available. Smaller puzzles such as four by four and six by six may be favored by children, beginners, and people with special needs; larger puzzles such as 12 by 12 and 16 by 16 may be favored by very proficient players who require additional challenge.

**Strategies for Solving Sudoku**

There exist many strategies for solving Sudoku. These depend on elimination, recognition of patterns, solving a simpler or more familiar problem, guess and check, and other methods that Polya describes. It is known that Sudoku belongs to a class of problems which is completely solvable by finite algorithm, known as “NP-complete.” (Seta & Yato, 2002). It is important to note that the algorithm which solves any given type of NP complete problem need not employable to humans; it may involve more steps than a human could accomplish in a lifetime, or more complexity than a human could manage. Fortunately, a powerful algorithm does exist, is known, and is feasible to
humans. This algorithm, based on methods similar to those described by Polya and incorporated into an iterative application of the pigeonhole principle, seems to have developed heuristically among Sudoku players and can be found in various popular and academic sources (See for example Aslaksen, 2002; Crook, 2009; Intelm, 2005; Stevens, 2007). Crook (2009) formally stated and proved this algorithm in a concise and highly readable article for the American Mathematical Society. Sudoku enthusiasts who use markups (also variously referred to as candidate lists, possibility matrices, etc, in Sudoku literature) are already familiar with this algorithm. Crook (2009) wrote about “preemptive sets”; these are already familiar to many Sudoku players under names such as naked pairs, hidden triples, swordfish, jellyfish, etc.

Aslaksen, a professor of Mathematics at the University of Singapore, and a judge of Sudoku contests in his free time, has presented lectures on this method, and published it as an extensive power point presentation on his website. “Intelm,” who writes under a pseudonym and says he is a retired high school math teacher, presented a method which is a large subset of Aslaksen's method in a popular e-book, How to Solve Every Sudoku Puzzle, distributed in electronic PDF format. In “A Pencil-and-Paper Algorithm for Solving Sudoku Puzzles,” Crook (2009) proved this algorithm.

Geostar Publications has given the researcher permission to reproduce, at no charge, How to Solve Every Sudoku Puzzle in PDF format in as many copies as necessary for the purpose of this study. Thus, How to Solve Every Sudoku Puzzle was offered as a resource for the participants to use if they wish. Another resource provided to the participants was Mastering Sudoku Week by Week by Paul Stevens (2007), a highly rated
book written for a general audience which explains the many of same methods (based on markups and proven by Crook) used by Aslaksen and Intelm. In addition, *Mastering Sudoku Week by Week* contains other traditional methods that have developed heuristically among Sudoku enthusiasts. Crook’s proof applies to these traditional strategies as well. *Mastering Sudoku Week by Week* was provided in addition to *How to Solve Every Sudoku Puzzle* because some participants in the preliminary investigation had indicated that *How to Solve Every Sudoku Puzzle* is too technical and difficult to read, while *Mastering Sudoku Week by Week* was preferred.

Cognitive Skills Which May be Exercised by Sudoku

Some of the cognitive skills which may be exercised by Sudoku include logical reasoning and flexibility in problem solving (Writer, 2007), lateral thinking (McCormack, 2005), and associative memory (Hopfield, 2006; see also Gurr 1987 and Skatsson, 2006). Perez (2007) suggested that Sudoku may exercise some of the same skills used in conducting chemistry experiments involving amino acids, and published variants on Sudoku that may be used in chemistry education.

Sudoku and Associative Memory

Sudoku is becoming popular in elementary and secondary education. Because it is enjoyable and engaging, it encourages practice leading to proficiency. (See also Bruner, 1972.) Sudoku may enhance cognitive function, in part, because it possibly exercises and strengthens associative memory. Associative memory, also known as content-addressed memory, is the usual way natural intelligence (that is, our minds) manipulate information for storage, retrieval, and problem solving. We think of
something, which reminds us of something else, which in turn reminds us of yet another idea. Through associative memory, humans can easily reconstruct a large body of information from a few clues, or make sense of meaningful information obscured by a great deal of noise. This is the principle behind the "CAPTCHA" tests that one frequently encounters when filling out a form online. (This is a contrived anagram for "Completely Automated Public Turing test to tell Computers and Humans Apart.") In this test, one is presented with a series of distorted letters, and then asked to type in the correct letters in a data entry field. Most people with normal eyesight and intelligence can do this fairly easily, but a computer program used to fill in websites to generate online mailing lists ("spam-bot"), usually fails at this task.

Content addressed memory seems so easy and natural to humans, yet it is an extremely challenging problem in artificial intelligence. In a paper published in 2008, "Searching for Memories, Sudoku, Implicit Check Bits, and the Iterative Use of Not-Always-Correct Rapid Neural Computation," J. Hopfield of Princeton University discussed how associative memory may be modeled in an excitatory-inhibitory network with implicit check bit patterns, and applied this model to the solution of a Sudoku puzzle by computer. Previous computer programs for solving Sudoku had relied on brute-force methods similar to those used by computers for creating Sudoku puzzles. Hopfield’s model solves a Sudoku puzzle by orderly pattern-recognition processes similar to those used by proficient Sudoku players. Hopfield even went so far as to suggest (p 3) that the "aha!" feeling that humans experience is “a form of check-bit validation,” and that the
sensation of discovering or creating is the brain's response when the correct information is constructed or re-constructed from minimal or even erroneous cues (Hopfield, 2008).

Hopfield's article, based on work he first published in 1982, is very technical and beyond the scope of this project. Skatsson (2006), reviewing Hopfield's article for ABC Science Online, wrote:

We all recognize the basic pattern of counting from one to nine, yet the task of completing a Sudoku puzzle is confounded because of the large number of possible permutations of this pattern. But every time we put the right number in the right place it provides us with a clue, which reduces the number of permutations. In this way Sudoku is based on a combination of logic and intelligent guesswork based on our abilities of associative memory. . . this may account for our strong feeling of "right" or "wrong" when we retrieve a memory from a minimal clue (2006)

Fortunately, Gurr (1987), in "Aha! Memory, Perception, Insight, and Problem Solving" made Hopfield's work more accessible. Gurr began by discussing the key role associative memory plays in appreciating humor, and how this relates to problem solving, memory, perception, insight, and the rewarding sensation of "aha!" that makes the struggle worthwhile.

Here, the researcher notes that perhaps we do not really need to discover, that is, independently innovate, in order to learn; instead, the resolution of an associative memory often feels like discovery. If this is true, then it is not necessary in education for students to reconstruct knowledge through self-directed inquiry in order to learn
effectively. Instead, it is sufficient for students to establish associations between what is known and what is to be learned. When these connections are sufficiently strong and stable, learners may feel like discoverers, confident in their knowledge. This feeling of discovery may serve as the “aha!” signal that the stable associations needed for meaningful learning have been established. Whether or not the student actually discovers (in the strict sense of the word) may be a moot point.

Associative memory could therefore become a very important paradigm in education. Through this paradigm, we could at last resolve the debate between those who favor inquiry-based learning, (which, though often highly effective, can be very expensive in terms of time and resources, and does not always yield results), and direct instruction, (which, though producing traditionally reliable results, has been criticized as “drill and kill” and can lead to compartmentalized learning). With this debate resolved, mathematics and science education in the United States could become more concerted and effective while remaining equitable, accessible, and affordable.

Availability and Accessibility of Sudoku

Sudoku is readily available in newspapers, magazines, books, and websites. Retired Judge Wayne Gould, who created the first commercially successful computer program to produce Sudoku puzzles, was rated one of Time Magazine's most influential people in 2005. Gould now provides daily puzzles to newspapers in 25 countries. A recent Google search on Sudoku turned up approximately 44,600,000 hits. A Google search for Sudoku puzzles in newspapers turned up approximately 3,520,000 hits, while a search for Sudoku puzzles in magazines turned up approximately 3,220,000 hits. A
search on Sudoku in education resulted in 12,200,000 Google hits. Dell Magazines, through Penny Publications, publishes 10 Sudoku periodical magazines, available worldwide at nominal cost. Also, there are, at present, over three thousand Sudoku books available on Amazon.com, in addition to many more titles in software programs and electronic devices. Thus, any educators who wish to incorporate Sudoku into their curricula would be readily able to find appropriate materials, at little or no cost, for a wide range of ability levels.

Summary and Conclusion

There are many effective methods for teaching problem solving. These methods are generally based on the work of Polya’s problem solving methods (Polya, 1945) originally developed for college students (Wilson et al, 1993). However, modern methods for teaching problem solving vary widely in order to address the needs of learners at varying ages and ability levels. The fit of the method to the needs of particular learners in a particular community is more important than the conformity of the method to any given theory or philosophy (Cotton, 2003). When learning complex material, metacognition and self-monitoring can be helpful, and journaling can support teacher and learner effectiveness (Delclos & Harrington, 1991; Liljedahl, 2007).

Educators as well as learners are often hindered by fear and anxiety associated with highly demanding subject material (Malinsky et al, 2006). In leisure activities and play, risks are minimized, repetition is rewarded, and learning can readily occur (Bruner, 1972). Although we cannot always rely on learners to discover “biologically secondary” knowledge for themselves (Genovese, 2003), we can rely on leisure activities and play to
support learning (Bruner, 1972). Puzzles are innately appealing to many humans, and are associated with advances in human culture (Danesi, 2002). Moreover, nonverbal puzzles such as Sudoku support visuospatial working memory, an important form of intelligence associated with mathematical ability and problem solving which can be improved by practice (Kyttälä & Lehto, 2008).

In particular, Sudoku is rich in cognitive benefits. Although solving a Sudoku puzzle in its modern form requires no arithmetic, Sudoku is certainly connected with mathematics. It is based on a construct invented by Euler (1776) as a study in combinatorics, and it can be understood as a problem in graph theory (Crook, 2009). As an NP-complete problem (Seta & Yato, 2002), it is solvable by finite algorithm which is, moreover, accessible to humans. This algorithm has been heuristically discovered by Sudoku enthusiasts (Aslaksen, 2008, Intelm, 2005, Stevens, 2007) and was recently proven to be mathematically valid (Crook, 2009).

Sudoku exercises many of the same cognitive abilities as are needed in mathematical problem solving (Writer, 2007, McCormack, 2005, Hopfield, 2006). It is important to note that these abilities can be improved by exercise (Kyttälä & Lehto, 2008). In particular, Sudoku exercises associative memory (Hopfield, 2006; Skatsson, 2006). Associative memory, though a very challenging problem in artificial intelligence, is a fundamental construct in natural intelligence (Hopfield 1982m 2008; Gurr, 1987). Associative memory may offer a new paradigm which may resolve many of the debates in education and re-direct the energy which has been devoted to these debates to a more effective and equitable education system.
As an educational resource, Sudoku offers many advantages. It is widely appealing, readily available at little or no cost, and accessible to people at various ages and levels of ability. Thus, the researcher finds it worthwhile to introduce Sudoku to college students and see if regular practice is associated with gains in cognitive fitness leading to improved mathematical problem solving ability.
CHAPTER 3 METHOD

This study is an action research piece, meaning that the researcher hoped to create change in her own setting and measure the extent of that change. No generalization beyond the researcher's domain is expected, since the researcher employed convenience sampling.

Participants were recruited from students at Central Washington University who were taking a freshman level mathematics class in the fall quarter of 2008 or the winter quarter of 2009. The recruitment process began with a brief presentation at the beginning of their regularly scheduled mathematics classes. In this presentation, the researcher explained the nature and purpose of the study to the potential participants. Pizza, Sudoku books, and certificates of participation were offered as incentive. Those who expressed interest were given slips of paper on which to put their contact information. (These slips of paper were promptly collected and placed in an envelope to protect confidentiality.) The researcher then contacted those who expressed interest, to invite them to an orientation meeting to be held on campus. Approximately 400 students viewed the initial presentation; of these, 38 expressed initial interest by giving their contact information. (In addition, this study was announced in the university newspaper, on the university website, and on fliers distributed around the university. No participants were recruited through these announcements.) Four orientation meetings were offered so that participants could attend whichever meeting was convenient to them. 14 participants attended an initial orientation meeting.
This attrition may be explained by considering that volunteers were recruited from different freshman mathematics classes and were not paid. Although many expressed initial interest in an activity which may help them academically, the practical difficulties of attending meetings outside of class time may have interfered. This attrition became even more pronounced in follow up. Even though only three of the original participants attended the follow up meetings for retesting, it was possible to obtain more information through email interviews.

At the orientation meeting, participants took Likert Scale surveys (Appendix A) developed by the researcher to measure their initial attitudes toward problem solving and participation in cognitively demanding leisure activities. They took the Klein test to measure initial visuospatial processing ability. The researcher had planned to re-administer these tests after treatment, and perform the Mann-Whitney U test on the results to determine if Sudoku practice had been associated with any changes in these variables. However, due to low attendance at the follow up meetings, the quantitative data analysis of the initial design of this experiment was not performed. Instead, this study relied on qualitative data gained from the journals provided by three participants who attended the follow up meeting, and on additional qualitative data obtained through email interviews with six of the participants who had attended the initial meetings.

Participants

The population was undergraduate students at Central Washington University in the fall quarter of 2008 or winter quarter of 2009, currently enrolled in a mathematics class, as required by their course of study.
The sample was a convenience sample of those who presented themselves as volunteers from the population. Of the initial participants who attended an orientation meeting, there were 10 females, 8 white and 2 non-white. There were four males, three white and one non-white. All were over 18 years old. Of the three participants who came for the follow up meetings, all were female; two white and one non-white. One of the white females indicated that she had Attention Deficit Hyperactivity Disorder. In initial testing, the participants scored average on the Klein Test, with a mean score of 14.25 and a range of 9 to 17. In the Likert scale surveys, all participants indicated that they already participated in various cognitively demanding leisure activities, and expressed generally positive attitudes toward problem solving. All participants except one indicated in their survey responses that they believed that problem solving is a skill that can be taught rather than “a natural talent that some people have and some people don’t.” (Interestingly, the one person who did believe this returned to the follow up meeting in order to report null results.) When surveyed about what grade they expected to receive in their current mathematics class, they expressed generally high expectations.

The participants were college students from the convenience sample. As this was a convenience sample, no restrictions were placed on gender, disability, socioeconomic background, or any other factor. (The only restriction on age was that participants under age 18 were not recruited). As indicated by their status as volunteers, their performance on the Klein Test, and their survey responses, these participants were likely more intelligent and motivated than average for the population of college students, although
there is no way to be certain of this. In addition, their initial attitudes toward problem solving and cognitively demanding leisure activities were positive from the outset.

**Apparatus**

Participants met at a computer lab on Central Washington University campus. Each participant had use of a desktop computer connected with the Internet. Using pencil and paper survey instruments developed by the researcher, participants were surveyed about their attitudes toward problem solving and participation in cognitively demanding leisure activities. (These surveys may be found in appendix A.) They also took the Klein Progressive Matrices Test, available online at [http://www.egopont.com](http://www.egopont.com), to measure nonverbal intelligence. They received copies of *Mastering Sudoku Week by Week* and folders containing survey instruments, journal sheets, and a CD with a PDF version of *How to Solve All Sudoku Puzzles* and links to various free online Sudoku resources.

**Procedure**

At the initial orientation meetings, there was at total of 14 participants. This was not enough for paired treatment and control, so the researcher planned to use pretreatment baseline measurements on the Likert scale surveys and Klein test as a control. Participants received copies of *Mastering Sudoku Week by Week* (Stevens, 2007) and *How to Solve All Sudoku Puzzles* (Intelm, 2005). Also, because the meeting had been scheduled for around meal time, light snacks were provided so that hunger would not impair participants’ performance on the Klein Test. (The researcher did not offer pizza before the testing and surveying out of concern that a heavy load on participants’ digestive systems would impair performance on the Klein Test. As a classroom teacher,
the researcher had often observed student lethargy in the class after lunch period. Also, the researcher was concerned that participants would take the pizza and leave without further participation.) Participants viewed a Power point presentation on this experiment so that they could give informed consent. After giving informed consent, participants used survey instruments developed by the researcher. Through these surveys, participants provided information about their expected grade in their current mathematics classes, participation in cognitively demanding leisure activities, and attitudes toward problem solving and cognitively demanding leisure activities. They took the Klein Test and recorded their raw scores, that is, number correct out of 18 questions attempted. After testing and surveying, pizza was provided.

Next, participants were encouraged to visit http://www.websudoku.com, a free interactive Sudoku website. Also a Sudoku puzzle from that website was displayed on the overhead screen. At this point it was hoped that participants would explore Sudoku and figure out how to play; instead, they quickly demanded to be shown some start-up strategies. This happened at each of the four orientation meetings. Therefore, the researcher showed them a basic start-up strategy familiar to many Sudoku players and variously known as “crosshatching,” “gridding,” or “hunting naked singles.” From that point, they became so engrossed in the puzzle that they virtually ignored the pizza. This ignoring of the food occurred at each of the four orientation meetings. After two hours, the participants were dismissed and encouraged to continue playing Sudoku and to record their experiences on the journal sheets which were provided.
Ten weeks later, at the start of the following quarter, participants were contacted and asked to attend a follow up meeting so that the surveys and Klein Test could be re-administered. Again, a choice of four meetings was offered so that participants could attend whichever meeting was convenient, and pizza was offered as an incentive. However, only three participants returned for these meetings, not enough for a data analysis. Therefore, these three participants were interviewed for case study. Additional interviews were conducted via email with six other participants who had come to the initial meetings. Enough information was gained from these interviews for a qualitative analysis.

HSR compliancy statement

Project used human subjects who have volunteered for this study. Researcher was trained in HSR requirements. This project was approved by Central Washington University HSR board.
CHAPTER 4 RESULTS

Although there was not enough data to perform quantitative analysis, enough information was gathered from participant journals, researcher journal, and email interviews to suggest an association between Sudoku practice and improved cognitive fitness leading to improved problem solving ability. These results will be reported in the form of three case studies and a compilation of interview results. (All names are changed to protect confidentiality of participants.)

Case 1: Linda, Chemistry Education

Linda, a Chemistry Education major, reported positive results from having practiced Sudoku. She kept an extensive journal of her efforts, showing a great deal of meta-cognition, that is, awareness of her evolving thought processes and strategies. She noted that having pre-conceptions as to “which numbers to seek first and why” was counterproductive, and resolved to take a more flexible approach. However, she felt that reading the Sudoku books provided was counterproductive after a certain point, and was no substitute for patience and practice. She found that it is helpful to work in a quiet area, free of distractions; on the other hand, working on a puzzle with another person can be effective also. These are all good study habits to encourage in her future students. As a prospective science educator, she said “Sudoku seems to be helping me to follow mechanisms in reactions in chemistry,” and recommended it to students in a lab where she served as a TA. (See Perez & Lamoureux, 2007.) With Sudoku, she said, she is learning that “a problem has a solution if only you don’t give up too soon.” This
perseverance inspired by confidence in the existence of a solution has carried over into her work in chemistry and chemistry education.

Linda reported that she got a C+ in Calculus III, half a letter grade higher than she expected. She attributed this to Sudoku practice, in particular to improved concentration and perseverance.

In her pre-treatment survey responses, Linda expressed favorable attitudes toward problem solving, and this may have been a factor in her benefitting from Sudoku practice. She disagreed strongly with statements such as “Problem solving is a natural talent; people either have it or they don’t” and “Some people can learn to solve problems and some people can’t.” On the other hand, she agreed strongly with statements such as “I like solving problems” and “I am eager to teach my students to solve problems.” On the Klein Test, she scored 15 prior to treatment and 17 after treatment.

Linda’s remarks and test scores confirmed the researcher’s belief that Sudoku may be useful for educators to improve their own problem solving skills as well as pass these skills along to their students. In a more controlled setting, such as incorporating Sudoku training into a mathematics class required for education majors, the benefits Linda that experienced could perhaps be multiplied many times over.

Case 2: Lucy, English Education

In contrast to Linda, Lucy demonstrated how static beliefs may be associated with whether one learns new problem solving skills. Like other participants, she said at the outset that she enjoys various cognitively demanding leisure activities. Unlike most other participants, Lucy agreed strongly with statements such as “Problem solving is a natural
talent; people either have it or they don’t” and “Some people can learn to solve problems and some people can’t.” It seems reasonable to suppose that someone who entertains such beliefs would be unlikely to benefit from any training effect, and this turned out to be true for her. Her initial score on the Klein test was 16 out of 20 and this did not change after treatment. She reported, “I did practice some Sudoku over the break but eventually forgot about it. When I tried again . . . it was harder to finish.” Thus, a difference appeared in these two case studies between Linda, who feel that problem solving is a skill which can be acquired, and Lucy, who felt that problem solving ability is a trait which one either has or lacks. Her final journal entry read simply “I can’t do this. My mind won’t compute.”

Like Linda, Lucy is a pre-service educator, in a position to influence the problem solving skills of future students. Her initial score on the Klein test was higher than Linda’s but her final score was lower. Perhaps, in a more structured situation with more peer support and external motivation, students like Lucy could benefit from Sudoku.

In fact, Lucy indicated that she would have liked to do Sudoku in groups with others. Five of the other initial 14 participants also indicated that they prefer to work in groups. (See also Vygotsky, 1978.) However, due to the difficulties of coordinating many volunteers who each had their own schedules and agendas, this was not feasible. Often, action research is carried out in a class that the researcher is teaching. This was not the case here. In order to conduct a meeting, it was necessary for the researcher to contact each participant individually by phone or email. Since caller-ID has become readily available, many people have a habit of screening their phone calls and answering
only calls from people they know. The researcher had email contact information on her participants as well; however, this was not always effective because people do not always read or respond to email from people they do not know. Occasionally a participant would respond to the researchers’ call or email and say that they were mistaken about the date of the meeting, or they forgot, or they had something other event that interfered, etc. As desirable as it would have been to schedule more meetings and opportunities for group work, this was not possible under these circumstances. In a more controlled situation, such as in a regular class, it would have been good to find ways to accommodate those who prefer collaborative learning, as well as reach those who are less receptive and improve their attitudes. Thus it would be possible to improve equity in education by addressing problem solving skills in a greater portion of the population.

Lucy reported a null result. Even so, her case is interesting because it shows how attitudes toward problem solving can be associated with performance.

Case 3: Erin, Biology

Erin is a Biology major who hopes to become a doctor. She said that she has Attention Deficit Hyperactivity Disorder. Erin reported many benefits from Sudoku practice. She got an A in her Calculus class, compared with her usual A/B performance in mathematics classes, and attributed this to improved concentration associated with Sudoku practice. She scored 17 out of 18 correct on the Klein test pre-treatment, and 18 correct post-treatment. She also wrote, “I have struggled with sleep problems for many years of my life and I notice that if I am having a hard time falling asleep I can do a few
Sudoku puzzles to calm my mind which allows me to fall asleep easily without using sleeping medication.”

Erin reported improved memory and confidence in her studies and attributed this to Sudoku practice. She wrote, “I am currently taking a biology anatomy physiology class that requires extensive memorization of anatomical structures which all need to be spelled correctly. Comparing my confidence on memorizing the material this quarter to last quarter I can definitely see a difference. I feel that this quarter I am able to memorize more material confidently and much faster than I could last quarter.” These benefits, of better sleep and easier memorization, were unexpected in that the researcher did not find them mentioned in the literature which was reviewed. However, these benefits are consistent with the overall improvement in cognitive fitness as found by Verghese et al in an older population.
Results, Email Interviews

Due to low attendance in follow up meetings, the researcher contacted all 14 of the original participants by email. In the email message, the researcher reminded the students of their part in the Sudoku study and asked them if they felt that they had benefitted from their participation in various ways. Thus, all participant responses are to be understood in the context of a discussion of possible benefits associated with Sudoku practice.

In the following discussion, interview questions are written in italics, and summary responses follows each question. Six participants responded to these email interviews. Not all of the respondents specifically answered all of the questions, and some answered by narrative rather than by specifically addressing the questions.

What grade did you get in your math class last quarter? Do you feel that Sudoku has helped in any way? If so, how?

Four responded that they had gotten an A; the other two did not respond to this question. One person who got A’s also remarked that “it took a lot of work” to achieve this; another said “with much studying I achieved an A.” In the following questions, participants went on to explain that they had experienced improved concentration, memory, perseverance, etc., in association with practicing Sudoku

Have you been practicing Sudoku? If so, how often? What levels can you do?

All six of the participants who replied stated that they had been practicing Sudoku, and had advanced in their Sudoku ability levels. One said she prefers Killer Sudoku, a variant which uses arithmetic as well as placement, and noted a system she
uses for markups using dots rather than numerals. (This system can be found in Sudoku literature and is sometimes favored by competitive Sudoku players because it saves time in creating the markups. It is more efficient but more demanding on working memory.) Another respondent said that she preferred the easy to medium puzzles, because they offer immediate gratification.

*Please read the next section, and tell me if you’ve noticed any improvement in yourself in these areas, or any other area you may have noticed. Sudoku has been said to enhance many cognitive skills. Among these skills are:*

**Short term or working memory capacity**

“I have definitely noticed improvement in my working memory capacity. I have had many tests to study for and have needed to retain a lot of information. I feel like I have been better able to recall my note cards that I make to help me study.”

“Lately it seems that I have been able to remember more information. When studying I have noticed my ability to remember more from my note cards in a shorter amount of time studying. For example, instead of studying for an hour a day in one subject, now I usually study for thirty minutes for that same subject while being able to acquire the same information.”

**Pattern recognition**

“I’ve had to learn different hormones and disorders in my classes and I have had an easier time finding similarities between disorders and their causes.”

“I can connect my thoughts better.”
“Yes, I have absolutely been able to connect ideas by patterns to come up with a conclusion. Recently taking tests I can think of one thing and it will remind me of another. This skill is very important during test times when I have a limited amount of time and can relate one question to another in the test and in result discover the answer.”

_Tolerance for ambiguity and frustration_

“Absolutely! I don’t give up as quickly… if I work on a problem that is challenging, and can get past the urge to just quit, I end up solving it soon after. It just takes a little extra will power, but typically I am successful.”

“When I play the easy puzzles I know it is set up to give me the necessary information to be able to solve the problem so I manage to always push through and finish. Just like when facing the day to day tasks I am faced with, I know that there is an answer and I strive to find it.”

_Attention span_

“I pay better attention in class and take good notes.”

“When I am in class I used to space out and let my mind wander. I have noticed now in class that I pay attention the whole time and take quality notes.”

_Lateral thinking_

“Yes, it (Sudoku) has helped with my problem solving . . . and coming up with new solutions and creative ideas!”

“Yes, I often think of different ways of solving everyday situations. . . . I often catch myself discovering new ways to solve problems. For example, I recently needed to
seal a patch in an exercise ball. I didn’t have the correct plug . . . (so) I successfully fixed the hole with foil, a piece of plastic, and a stick of gum.”

*Flexibility in problem solving*

“I sometimes like to race my boyfriend and we both solve them so differently, but end up finishing the around the same time. We like to compare what spots got us stuck and where we started. I think this has showed me that there is typically more than one right answer to problems or more than one correct approach.”

“I find myself having to come up with new ideas and paths to solve problems. I have learned that there are many ways to solve everyday problems.”

*Other comments*

“Thank you for showing me how to solve Sudoku puzzles and the acquired abilities that come with this fun game.”

“I can feel my brain stretching!”

“I have really enjoyed learning and playing Sudoku! Thank you for teaching me!”

“I can definitely see a difference!”
Discussion

The participants who remained in the study noted many of the same benefits from Sudoku as those cited in Chapter 2, and other benefits besides. If the adult brain is indeed as dynamic as recent research suggests, Sudoku seems to be a worthwhile activity to enhance cognitive fitness leading to improved problem solving ability and academic performance. As a form of play, it encourages practice leading to proficiency. (Bruner J. 1972). As a puzzle, it enjoys universal appeal (Danesi, 2002). As a non-verbal puzzle, it may exercise some of the same types of non-verbal intelligence demanded in mathematics; moreover, these abilities can be improved by exercise (Kyttälä & Lehto 2008). Although it is still not certain as to exactly how Sudoku practice may improve cognitive fitness to support mathematical problem solving education, these results suggest that there were some sort of benefits that may be replicable and applicable to other mathematics learners.

A major drawback of this study was in the management of individual volunteer participants. Often studies like this are carried out in connection with a class that the researcher is teaching. The students in that class become a readily manageable sample, so that data analysis can be performed. Sometimes research is funded by a grant, so that volunteers can be paid to participate. This researcher was not teaching a regular class. She had to recruit volunteers individually, keep them engaged out of her own resources, and could not pay them. In any study of human participants, the Pygmalion and Hawthorne effects must be considered, particularly in such a small sample.
CHAPTER 5 SUMMARY CONCLUSION AND RECOMMENDATIONS

Summary and Conclusion

Participants who remained in the study, when interviewed about their Sudoku practice and their subjective experience of various cognitive benefits in the context of Sudoku practice, reported many benefits and no negative effects. They reported improved performance in their current mathematics classes and improved problem solving ability. Components of this improved problem solving ability include working memory, pattern recognition, tolerance for ambiguity and frustration, attention span, lateral thinking, and flexibility in problem solving. They also expressed pleasure in the accomplishments they had experienced through participation in this study.

Recommendations

Although no quantitative analysis was possible, these qualitative results suggest that it would be worthwhile to incorporate Sudoku into a mathematics class which includes problem solving.

The participants were self-selected volunteers, therefore, they were probably more motivated and possibly more intelligent than the general population. It would be especially interesting to see if the gains they experienced could also be achieved in a larger group of students with varying levels of ability and motivation. In such a setting, working in groups would be very helpful. See also Vygotsky (1978).

An early hypothesis in the development of this project was that Sudoku may provide a threshold experience to stimulate the development of Stage 4 thinking. This hypothesis was discarded because of the practical difficulties of testing under the
researcher’s limited circumstances. However, it would be a worthwhile hypothesis to investigate in a further study.

The researcher intends to continue using Sudoku and other puzzles and games to encourage her future students to improve their cognitive fitness and problem solving ability. Sudoku is associated with many gains in cognitive fitness in those who persevered. Therefore, it would be worthwhile to introduce it to a larger and more organized group and see if any or all of these gains could be generalized. The original design of this study, with Likert Scale surveys, journaling, and use of the Klein Test would be feasible and worthwhile in a controlled setting such as a regular classroom. The researcher could carry out such a study later on in her career, perhaps as a journal article for publication or as a PhD dissertation.
REFERENCES


Hopfield, J. (1982) Neural networks and physical systems with emergent collective computational properties. Procedures of the National Academy of Sciences


Survey -- Attitudes Toward Problem Solving

Please give your honest opinion. Please do not sign your name. This survey will not affect your grade or status at Central Washington University.

1. I like solving problems.
   | DISAGREE | NO OPINION | AGREE |
   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

2. Problem solving is a natural talent; people either have it or they don't.
   | DISAGREE | NO OPINION | AGREE |
   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

3. I am eager to teach my students to solve problems.
   | DISAGREE | NO OPINION | AGREE |
   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

4. I would rather not have to worry about teaching anyone to solve problems.
   | DISAGREE | NO OPINION | AGREE |
   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

5. Some people can learn to solve problems and some people can't.
   | DISAGREE | NO OPINION | AGREE |
   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

6. It is important that all students have a chance to learn to solve problems.
   | DISAGREE | NO OPINION | AGREE |
   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
More About You... NAME__________________________

This survey is to find out your attitudes toward cognitively demanding leisure activities and your expectations about your current math class based on your previous experience. We need your name for this survey, so that we can coordinate meetings of this group, and so that we can observe how these attitudes may affect your experience in this Sudoku study, and how your experience in this Sudoku study may affect your attitudes. These surveys will be kept in a secure location and destroyed at the end of this study. They will not affect your grades or status at CWU in any way.

Would you like to work on Sudoku in groups with other people in this study?  
Yes  No

If yes, what times and days is best for you to meet with other members of this study?

Please answer the following questions.  
Do you like to do any sort of puzzles such as crossword, Sudoku, Tetris, Rubik Cube, etc?  Yes  No

Do you like to play any sort of thinking games such as bridge, chess, backgammon, etc?  
Yes  No

Do you like to read mystery novels?  
Yes  No

Do you like "who dun it" shows such as CSI  
Yes  No

Based on your previous experience in math class, what grade do you expect to receive in the mathematics class you are currently taking?  A  B  C  D  F
On Magic Squares
Leonard Euler

§1. It is customary for a square to be called a magic square when its cells are inscribed with the natural numbers in such a way that the sums of the numbers through each row, column and both diagonals are mutually equal. Then, if the square were divided into x equal parts, there would be xx cells altogether, and each of the rows, columns and both diagonals would contain x cells, in which each one of the natural numbers 1, 2, 3, 4, . . . , xx would be arranged, such that the sums for all these lines would be equal to each other. For this, the sum of all the numbers from 1 to xx is

\[ \frac{xx(1 + xx)}{2} \]

and the sum for each line is equal to

\[ \frac{x(1 + xx)}{2} \]

by which, for x = 3 the sum for a single line would be equal to 15.

§2. Thus into however many cells a square is divided, the sum of the numbers deposited in each line can be easily calculated, from which the sums for all the lines of each square can themselves be easily calculated.

Delivered to the St. Petersburg Academy October 17, 1776. Originally published De quadratis magicis, Commentationes

Annotators note: This copy of Euler’s De quadratis magicis was translated from the original Latin by Jordan Bell of Carleton University, Ottawa, Ontario, Canada. It has been further annotated for clarity to the modern reader by J. Pinkney, graduate student in the Master of Arts in Mathematics Education program at Central Washington University, 2009. Bell’s translation of Euler’s original work is in the left column; notes are in the right column.

Typesetters note: on occasion the type is changed from 12 pt to 10 pt in order to align annotations as closely as possible beside Euler’s text.

Notes:

§1 Euler wrote xx as we would write \( x^2 \)

§2 The sum for the entire square would be x times the sum for each “line” (row, column, or diagonal); thus, with \( x = 3 \), we have

\[ \frac{9(1 + 9)}{2} = 45 \]

For example, in a 3x3 square the sum for each row, column, or major diagonal would be \( \frac{3(1 + 9)}{2} = 15 \), as in this example:

1 7 4  \hspace{1cm} 1 + 7 + 4 = 15
8 5 2  \hspace{1cm} 8 + 5 + 2 = 15
6 3 9  \hspace{1cm} 6 + 3 + 9 = 15

for the rows. Also, the sum for each column and major diagonal is 15.
§3. To help us investigate a certain rule for forming magic squares of all orders, it is very interesting to observe that all the numbers 1, 2, 3 to \( xx \) can be represented with this formula: \( mx + n \),

for if we successively have \( m \) take all the values 0, 1, 2, 3, 4 to \( x-1 \), and then \( n \) take all the values 1, 2, 3, 4, \ldots, \( x \), it is clear that all the numbers from 1 to \( xx \) can be represented by combining each of the values of \( m \) with each of the values of \( n \).

Furthermore, all the numbers to be inscribed on the square with the formula \( mx + n \) are able to be expressed using two parts, always in this order, where we use the Latin letters \( a, b, c, d \) etc. for the first part \( mx \), and the Greek letters \( \alpha, \beta, \gamma, \delta \) etc. for the second part \( n \), where it is clear that for any number \( x \), there is always a Latin and Greek letter that can be equal to \( x \) by having the Latin letters be \( 0x, 1x, 2x, 3x \) to \( (x-1)x \) and the Greek letters take the values 1, 2, 3, 4, \ldots, \( x \).

Notes:

§2 In figuring out where to place each number, the first thing we need to do is decide which number goes in the middle cell. In the case of a 3x3 square, this number is always 5. In section 8, below, Euler showed how to calculate this.

§3 First, Euler discussed a variant on what we now know as the “remainder theorem,” which will be a key idea to these calculations. In considering the numbers from 1 to \( x^2 \) for any given \( x \) the quotient can be at most \( x \) and the remainder can be at most \( x - 1 \), as we normally write. However, Euler wrote this differently. He let \( m \), the quotient, take on values 0 to \( x-1 \), and \( n \), the remainder, takes on values 1 to \( x \). The reason for this will become clear when we see how Euler used permutations on the quotients and remainders to construct his magic squares.

Thus, the Latin letters will represent multiples of \( x \), while the Greek letters will represent remainders.
However, this ordering of the Latin and Greek letters is not fixed, and any Latin letter can denote 0x, 1x, 2x etc., as long as all the different values are taken by them, with the same holding for the Greek letters

\[ x \quad \text{xx} \quad \frac{x(1+\text{xx})}{2} \]

<table>
<thead>
<tr>
<th>( \text{xx} )</th>
<th>( x \frac{x(1+\text{xx})}{2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 1</td>
<td>1</td>
</tr>
<tr>
<td>2 4</td>
<td>5</td>
</tr>
<tr>
<td>3 9</td>
<td>15</td>
</tr>
<tr>
<td>4 16</td>
<td>34</td>
</tr>
<tr>
<td>5 25</td>
<td>65</td>
</tr>
<tr>
<td>6 36</td>
<td>111</td>
</tr>
<tr>
<td>7 49</td>
<td>175</td>
</tr>
<tr>
<td>8 64</td>
<td>260</td>
</tr>
<tr>
<td>9 81</td>
<td>369</td>
</tr>
</tbody>
</table>

Furthermore, all the numbers to be inscribed on the square with the formula \( mx + n \) are able to be expressed using two parts, always in this order, where we use the Latin letters a, b, c, d etc. for the first part \( mx \), and the Greek letters α, β, γ, δ, etc. for the second part \( n \), where it is clear that for any number \( x \), there is always a Latin and Greek letter that can be equal to \( x \) by having the Latin letters be 0x, 1x, 2x, 3x to \( x-1 \)x and the Greek letters take the values 1, 2, 3, 4, ..., \( x \). However, this ordering of the Latin and Greek letters is not fixed, and any Latin letter can denote 0x, 1x, 2x etc., as long as all the different values are taken by them, with the same holding for the Greek letters.

Notes:

. In the construction to follow, \( x = 3 \), \( m \in \{0,1,2\} \) and \( n \in \{1,2,3\} \) the numbers 1 to 9 will be represented as follows:

\[
\begin{align*}
1 &= (0)3 + 1 \\
2 &= (0)3 + 2 \\
3 &= (0)3 + 3 \\
4 &= (1)3 + 1 \\
5 &= (1)3 + 2 \\
6 &= (1)3 + 3 \\
7 &= (2)3 + 1 \\
8 &= (2)3 + 2 \\
9 &= (2)3 + 3
\end{align*}
\]

“This order of the Latin and Greek letters is not fixed,” that is, the letters \( a, b, c \) may take on any values from the set \{ 0, 3, 6 \}, in no particular order.

For example, \( a \) need not represent 0; \( a \) may represent 3 or 6. Etc.

In this system, the letters will first be placed, then their values will be calculated, somewhat like breaking a cipher.
§4. Now, any number that is to be inscribed on the square can be represented with a pair of a Latin and a Greek letter, say by $b + \delta$ or $a + \beta$, etc.; that is, each number can be represented with these two parts. If each of the Latin letters are joined together with each of the Greek letters, it is clear that all the numbers from 1 to xx should result, and it is also clear that each different combination of letters always produces different numbers, with no number able to be repeated.

§5. Therefore all the numbers are able to be represented by the combination of a Latin and Greek letter. This in fact yields a rule for the construction of magic squares.

First, the Latin letters are inscribed in every cell of the square so that the sum for all the lines are the same, where if there were $x$ letters there would be $xx$ cells altogether in the square, and it is clear that each letter would be repeated $x$ times.

Similarly, the Greek letters are then inscribed in all the cells of the square, such that the sums for all the lines are equal.

Then, for all the lines the sums of these numbers made by a combination of a Latin and Greek letter would be equal.

Furthermore, in an arrangement where every Latin letter is combined with every Greek letter, with this method none of the numbers from 1 to $xx$ would be missed, and neither would any of them occur twice.

Notes:

§4. $x =$ the number of rows or columns in the Latin square. We are considering a 3x3 square with 9 cells, so $x = 3$

Each Latin letter $a$, $b$, $c$, etc represents a multiple of $x$ from 0 to $x(x-1)$. For $x = 3$, $a$, $b$, and $c$ represent the numbers 0, 3, 6 (though not necessarily in that order). Also, each Greek letter $\alpha$, $\beta$, $\gamma$, $\delta$, etc represents a number from 1 to $x$, so for $x = 3$, $\alpha$, $\beta$, $\gamma$ represent 1, 2, and 3 (not necessarily in that order.) By permuting the Latin and Greek letters we can uniquely represent each number from 1 to 9. The result has all numbers 1 to 9 in a 3x3 square; this is more complex than a 3x3 Sudoku, in which only the numbers 1, 2, and 3 would appear.

§5 For creating a Sudoku, we can use a similar process only we do not need the Greek letters. Using only the Latin letters will be enough. However, it is not trivial to figure out where to place these letters. Systematic ways to arrange the letters so that each occurs once in each row, column, and major diagonal will be demonstrated next. A summary follows:

First the top row is filled in, with the entries arbitrary. Next, one diagonal is filled in, so that no value is repeated in a row or column. (For a 3x3 square there is only one way to do this once the top row has been filled in. For larger squares there are more choices.)

The process for filling in the rest of the square depends on an axis of symmetry, which in turn depends on whether $x$, the number of rows and columns, is odd or even.

For a square with an odd number $x$ of rows and columns, the center column is the axis of symmetry. For a square with an even number $x$ of rows and columns, the left main diagonal is the axis of symmetry. There are choices to be made in filling in the top row and perhaps in filling in the diagonal. The values of many of the remaining positions will be determined by these choices, as we will see. For Euler’s process of creating a Latin square, the Greek letters are placed next, in a systematic way that is determined by the placement of the Latin letters. Finally, their numeric values are calculated to create the Latin Square with each of the numbers from 1 to $x^2$ occurring exactly once in the entire grid and the sum along each row, column, and diagonal equal.
§6. For using this rule to make each type of square according to how many cells it has, it is clear right away to start with nine cells, because a square with four cells does not have enough room for such an arrangement. Furthermore, in general it is seen that for each type there are \( x \) Latin and Greek letters, and that all the lines have the same number of cells, with the given conditions satisfied if each line has all the Latin and Greek letters inscribed in it. However, if the same letter occurs two or three times in some line, it is always then necessary to have the sum of all the letters occurring in each line equal to the sums of all the Latin letters \( a + b + c + d + \) etc. or Greek letters \( \alpha + \beta + \gamma + \delta + \) etc.

Notes
§6 A 3x3 Latin Square would have the numbers 1 to 9 arranged so that the sum along each row, column, and major diagonal is 15, and this is done by permuting the Latin letters \( a, b, c \) and the Greek letters \( \alpha, \beta, \gamma, \delta \) then calculating their numeric values.

It is also possible to create a Latin square of this type, with each number occurring exactly once and with some letters re-occurring in a row or column; this is explored in §20 when Euler introduced a variant on his method to create a 4x4 Latin square.
I. Types of squares divided into 9 cells

§7. For this type, it follows that \( x = 3 \), and we have the Latin letters \( a, b, c \) and the Greek letters \( \alpha, \beta, \gamma \) where the Latin letters have the values 0, 3, 6 and the Greek letters 1, 2, 3. We now begin with the Latin letters \( a, b, c \) and it is easy to inscribe them in the 9 cells of our square such that in each row 3 and column all three letters occur. For instance, this figure can be seen:

\[
\begin{array}{ccc}
  a & b & c \\
  b & c & a \\
  c & a & b \\
\end{array}
\]

where now too in one diagonal each of the three letters \( a, b, c \) appears, but in the other the same letter \( c \) is repeated three times; it is easy to see that it is not possible to have all the three different letters in both of the lines at once.

However, this does not cause a problem as long as the diagonal \( 3c \) is equal to the sum of the other diagonal \( a+b+c \); that is, providing that \( 2c = a+b \).

Notes:

§7 The Latin letters take on values corresponding to multiples of the base \( x \), and the Greek letters take on values corresponding to the numbers 1 through \( x \) which will be used as remainders. This may seem strange until one gets used to the idea that \( x \) itself may be a remainder. For example, 6 is expressed as \((1\times3) + 3\) rather than as \(2\times3\). However, when we see how Euler uses this system to place the numbers in the grid, the merit of this notation will become apparent.

The combination of \( c \) with the Greek letters representing the remainders will generate three different numbers in the right diagonal. This recurrence of letters in the right diagonal prefigures the square we will see in §20. There, a 4x4 square which will have repeated Latin letters in every row, yet it will satisfy the condition of no number repeated and the same sum occurring for each row, column, and diagonal.
From this, it is clear that $c$ should be taken to be 3, and the letters $a$ and $b$ assigned the values 0 and 6; thus it would be $2c = a + b$. Hence it would be possible to have either $a = 0$ or $b = 0$, and from this, the sum of each line results as $a + b + c = 9$.

§8. Similarly, the Greek letters can be distributed in a square of this type, and we can represent them in this figure with an inverse arrangement:

\[
\begin{array}{ccc}
\gamma & \beta & \alpha \\
\alpha & \gamma & \beta \\
\beta & \alpha & \gamma \\
\end{array}
\]

for which it is necessary to have $2\gamma = \alpha + \beta$ and thus $\gamma = 2$.

Then, if we combine in a natural way each of the cells from the first figure with each of the cells from the second figure, it will be seen that each of the Latin letters is combined with each of the Greek letters, such that from this combination all the numbers from 1 to 9 result; this would produce the following figure:

\[
\begin{array}{ccc}
a\gamma & b\beta & c\alpha \\
b\alpha & c\gamma & a\beta \\
c\beta & a\alpha & b\gamma \\
\end{array}
\]

where it is noted that two letters being joined together does not mean the product, but instead denotes a combination.

§9. With it taken in this figure that $c = 3$ and $\gamma = 2$, then the letters $a$ and $b$ can be assigned 0 and 6, and also the letters $\alpha$ and $\beta$ the values 1 and 3. If we suppose that $a = 0$ and $b = 6$, and that $\alpha = 1$ and $\beta = 3$, the following magic square will be seen:

\[
\begin{array}{ccc}
2 & 9 & 4 \\
7 & 5 & 3 \\
6 & 1 & 8 \\
\end{array}
\]

Where the sum for each line is 15.

Notes:
- That is, $a + b + c = 3c$
  
  \[
  0 + 3 + 6 = 3c \\
  9 = 3c \\
  3 = c 
  \]
  
  So $c$ must be 3; however $a$ can be 0 or 6 with $b$ taking on the other value.

§8 “with an inverse arrangement” that is, with the Latin letters

\[
a\ b\ c \\
b\ c\ a \\
c\ a\ b \\
\]

reflected across the center column and replaced with their Greek counterparts, $a \rightarrow \alpha$, $b \rightarrow \beta$ and $c \rightarrow \gamma$.

Now, since $\alpha$, $\beta$, and $\gamma$ represent elements of \{1, 2, 3\} we must have $\alpha + \beta + \gamma = 3\gamma$ so $\gamma = 2$.

“Then, if we combine in a natural way…” that is, by overlaying the Greek and Latin letters, we obtain the figure shown.

“denotes a combination,” that is, addition. For example, $a\gamma$ means $a + \gamma = 0 + 2 = 2$.

§9 So in this square we have

\[
a = 0. \ b = 6, \ c = 3 \text{ and} \\
\alpha = 1, \ \beta = 3, \ \gamma = 2.
\]

Replacing these letters with their number values, and adding the Greek and Latin letters which appear together in a cell, we obtain the square shown.

For example, in the first row,

\[
a\gamma = 0 + 2 = 2 \\
b\beta = 6 + 3 = 9 \\
c\alpha = 3 + 1 = 4 \\
\]
If we permute the values of the letters $a$ and $b$, and likewise $\alpha$ and $\beta$, it is clear that a different square will be seen.

§10. It is clear that this is a sufficient arrangement of Latin and Greek letters, and of particular importance in this is the placement such that each Latin letter is combined with each Greek letter, and in our arrangement this seems to have occurred by coincidence.

So that we do not have to leave this to coincidence, before proceeding we observe that the arrangement of the Greek letters $\alpha$, $\beta$, $\gamma$, does not depend on the arrangement of the Latin letters $a$, $b$, $c$.

Thus for each line, it could be set that the Greek letters are combined with their Latin equivalents, e.g. $\alpha$ with $a$, $\beta$ with $b$ and $\gamma$ with $c$.

Hence the first row could be set $a\alpha$, $b\beta$, $c\gamma$, and since the same Greek letter would not occur twice in any row or column, it can simply be for the second row $b\gamma$, $c\alpha$, $a\beta$, and for the third $c\beta$, $a\gamma$, $b\alpha$, from which this square results:

\[
\begin{array}{ccc}
a\alpha & b\beta & c\gamma \\
b\gamma & c\alpha & a\beta \\
c\beta & a\gamma & b\alpha
\end{array}
\]

where, because in the left diagonal the Greek letter $\alpha$ occurs three times, it is necessary that $3\alpha = \alpha + \beta + \gamma$ that is, $2\alpha = \beta + \gamma$ which then determines the value of $\alpha$, namely $\alpha = 2$.

In the way we see that $c = 3$. However, this does not make a new magic square.

Notes
Now we assign $a = 6$, $b = 0$, $c = 3$ and $\alpha = 3$, $\beta = 1$, $\gamma = 3$ and obtain

\[
\begin{array}{ccc}
8 & 1 & 6 \\
3 & 5 & 7 \\
4 & 9 & 2
\end{array}
\]

This is a reflection of the square in section 9 through its center cell. Note that 5 remains fixed, as shown in section 2.

§10
The Greek symbol in the middle cell must take the value 2, and the Latin symbol in the middle cell must take the value 3, as before. As Euler noted, “this does not make a new square.” To see this, consider the diagonal from upper right to lower left. Substituting, we obtain $c\gamma = 3+1 = 4$; $c\alpha = 3+2 = 5$; $c\alpha = 3 + 3 = 6$. This gives the same square as before:

\[
\begin{array}{ccc}
2 & 9 & 4 \\
7 & 5 & 3 \\
6 & 1 & 8
\end{array}
\]
§11. Although in this first type the arrangement of the Greek letters is not difficult to carry out, for squares with larger numbers of cells it is useful to give a method by which to inscribe the Greek letters after the Latin letters have been deposited.

For this, a line is chosen in the middle of the rows, columns or diagonals, such that on either side of the line, the cells that are equally far away contain different Latin letters. For instance, in this case the middle column is taken, around which in the first row are the letters a and c, in the second b and a, and in the third c and b, such that two different letters are always on each side.

§12. Then after such a middle line has been chosen, in it each Latin letter is combined with its Greek equivalent, and then on either side of this, the Greek letters are placed with their reflected equivalents; for instance, here in this way such a figure results:

\[
\begin{array}{ccc}
\alpha & \beta & \gamma \\
\beta & \gamma & \alpha \\
\gamma & \alpha & \beta \\
\end{array}
\]

in which we have clearly combined all the Latin letters with all the Greek letters. Then, so that the conditions can be satisfied for the diagonal, we take that \(2c = a + b\) and \(2\gamma = \alpha + \beta\).

This figure is in fact not different from the one which we made earlier in §8. Also, it can be seen that no matter how the rows and columns are permuted, the sums for the rows and columns are not changed.

Notes:

§11 With larger grids it is convenient to introduce a notation for referring to specific cells. So the usual matrix notation will be used, with \((r,c)\) referring to the cell in the \(r\) row and the \(c\) column. For example, \((2,3)\) refers to the cell in the second row, third column.

Here the idea of an axis of symmetry is introduced. This axis will be the center column for squares with odd \(x\), and the center diagonal for squares with even \(x\).

§12. On the center column, the axis of symmetry, the Latin / Greek equivalents are combined as follows: \(a \rightarrow \alpha\), \(b \rightarrow \beta\), \(c \rightarrow \gamma\), \(d \rightarrow \delta\), \(e \rightarrow \varepsilon\), etc. Then, these equivalents are reflected across the center column:

\[
\begin{array}{ccc}
\alpha & \beta & \gamma \\
\beta & \gamma & \alpha \\
\gamma & \alpha & \beta \\
\end{array}
\]

“so that the conditions can be satisfied for the diagonal” that is, the diagonals must add to the same number as the sum for rows and columns. Thus, from the right diagonal, we have \(3c = a + b + c\), and from the left diagonal we have \(3\gamma = \alpha + \beta + \gamma\).
However, for the diagonals this can make a very large difference; if the first column were taken and put on the other end, this figure would be seen:

\[
\begin{array}{ccc}
  b\beta & c\alpha & a\gamma \\
  c\gamma & a\beta & b\alpha \\
  a\alpha & b\gamma & c\beta
\end{array}
\]

where for the diagonals it must be taken

\[
2a = b + c \text{ and } 2\beta = \alpha + \gamma
\]

for which it is noted that everything has been transposed, which is an observation that will be very helpful for the following types.

II. Types of squares divided into 16 cells

§13. Here it is \( x = 4 \), and we have the four Latin letters \( a, b, c, d \), with the values \( 0, 4, 8, 12 \), and also the four Greek letters \( \alpha, \beta, \gamma, \delta \) with the four values \( 1, 2, 3, 4 \). Therefore we first inscribe these Latin letters in the square, such that in each row and column all the four letters occur, and if it is possible, to have this in both diagonals also.

§14. Since there is no prescribed arrangement for these letters \( a, b, c, d \), in the first row we inscribe them in order, and for the left diagonal, in the second cell of this diagonal we place either the letter \( c \) or \( d \). If we were to write \( c \), all the other letters would then be determined, providing that it is made sure that the same letter is not written twice in any row or column. We form the following figure in this way:

\[
\begin{array}{cccc}
  a & b & c & d \\
  d & c & b & a \\
  b & a & d & c \\
  c & d & a & b
\end{array}
\]

where each diagonal contains all four letters, so that no conditions are prescribed for the values of the letters \( a, b, c, d \).

Notes

The resulting square is:

\[
\begin{array}{ccc}
  4 & 9 & 2 \\
  3 & 5 & 7 \\
  8 & 1 & 6
\end{array}
\]

which compare

\[
\begin{array}{ccc}
  2 & 9 & 4 \\
  7 & 5 & 3 \\
  6 & 1 & 8
\end{array}
\]

§13. Now with \( x = 4 \) we construct a 4x4 Latin square. The Latin letters will denote multiples of 4, while the Greek letters will denote the remainders:

\( 1, 2, 3, 4 \).

The placement of letters in the top row can be arbitrary. Here, Euler began by placing them in their usual alphabetical order. Next he made a choice whether to place \( c \) or \( d \) in the cell in position (2,2). Choosing to place \( c \) in this cell, along with the condition that each value may occur exactly one in each row and column, determines the placement of the other values along the left diagonal. If we instead place \( d \) in (2,2) the following square would result:

\[
\begin{array}{cccc}
  a & b & c & d \\
  c & d & a & b \\
  d & c & b & a \\
  b & a & d & c
\end{array}
\]

It may not be immediately apparent that the choice of a value for (2,2) forces all of the other entries in the diagonal but a bit of experimentation verifies this. Also, if we were to write the letter \( d \) in (2,2), the resulting figure would be the only other different possible arrangement. Thus, with this figure all the other possibilities would be considered.
§15. Now, for inscribing the Greek letters, since no middle row or column is given, we take the diagonal $a, c, d, b$ as the center, and we find at once that in the cells equally far away on either side the two letters are distinct, from which it is seen that the rule given before in §11 can be used. Therefore, first we combine the letters in this diagonal with their Greek equivalents, and then combine the Greek letters with their reflected equivalents; in this way, the following figure is formed:

\[
\begin{array}{cccc}
  a\alpha & b\delta & c\beta & d\gamma \\
  d\beta & c\gamma & b\alpha & a\delta \\
  b\gamma & a\beta & d\delta & c\alpha \\
  c\delta & d\alpha & a\gamma & b\beta \\
\end{array}
\]

§16. Thus in this figure, all the four Latin and Greek letters occur in all the rows, columns and the full diagonals, and because of this the four values of these letters can be set without any restrictions. Since there are 24 variations of four letters, altogether 576 different figures can be formed, and a good many of the ones made in this way have structures that are mutually different.

§17. By no means should it be thought that all types of magic squares can be made according to this figure. There are many others that can be made, where each row does not contain all four Latin and Greek letters, and that nevertheless fulfill the prescribed conditions.

Notes:
When $x$ is even, the left diagonal rather than the middle row is used as an axis of symmetry (first introduced in §11) for the systematic arrangement of Greek and Latin letters.

“…with their Greek equivalents”: that is, along the left diagonal we get $a\alpha$ in $(1,1)$, $b\beta$ in $(2,2)$, $c\gamma$ in $(3,3)$ and $d\delta$ in $(4,4)$.

The “reflected equivalent” of $b$ is $\delta$ because $b$ and $d$ are on opposite sides of the diagonal and $\delta$ is the Greek equivalent of $d$. Thus, $b\delta$ goes in $(1,2)$ and $d\beta$ goes in $(2,1)$. Also, $c\beta$ goes in $(1,3)$ while $b\gamma$ goes in $(3,1)$, and so on. In this way the entire grid can be filled in.

There are 24 ways to place the four Latin letters in the top row, and 24 ways to place the four Greek letters in the top row. These are independent. Also, there are 2 choices for the value in $(2,2)$, which then fixes the other choices.

So in the construction of a Sudoku we could begin by placing elements in the top row. Then we fill in the axis of reflective symmetry (the middle column for an odd number and the left diagonal for an even number.) These entries would serve as “guides” From these we could deduce the placement of the other elements in the grid.
Some of these can be made by transposing columns or rows; for instance, if in the above figure the first column is put at the end, this figure will be seen:

\[
\begin{array}{cccc}
  b\delta & c\beta & d\gamma & a\alpha \\
  c\gamma & b\alpha & a\delta & d\beta \\
  a\beta & d\delta & c\alpha & b\gamma \\
  d\alpha & a\gamma & b\beta & c\delta
\end{array}
\]

§17b

where indeed in each row and column all the Latin and Greek letters still appear, but where in the left diagonal, descending to the right only two Latin letters occur, namely b and c, and in which also is only a pair of Greek letters, \(\alpha\) and \(\delta\).

In the other diagonal are the other two Latin letters a and d, and as before, the Greek letters \(\alpha\) and \(\delta\).

§18. So that this figure satisfies the prescribed conditions, each letter can take no more than a single value, which suggests the restriction for the Latin letters of:

\[
b + c = a + d,
\]

and similarly for the Greek letters that:

\[
\alpha + \delta = \beta + \gamma
\]

and so, if we take \(a = 0\), then it follows at once \(d = 12\), and that \(b = 4\) and \(c = 8\), or vice versa, that \(c = 4\) and \(b = 8\).

Similarly for the Greek letters, if we take \(\alpha = 1\), then it must be \(\delta = 4\), and then \(\beta = 2\) and \(\gamma = 3\). From this, a magic square determined as such is made:

\[
\begin{array}{cccc}
  8 & 10 & 15 & 1 \\
  11 & 5 & 4 & 14 \\
  2 & 16 & 9 & 7 \\
  13 & 3 & 6 & 12
\end{array}
\]

where clearly the sum for each line is 34.

Notes:

§17b. That is, column 1 moves to column 4 and the other three columns shifted over one space to the left. So \(a\alpha\) is moved from (1, 1) to (1, 4); \(d\beta\) is moved from (2, 1) to (2, 4), and so on. \(b\delta\) is moved from (2, 2) to (2, 1) and all of the other entries in the grid are moved form \((r, c)\) to \((r, c-1)\) for \(c > 1\).

When we move column 1 to column 4 and shift the other three columns over by one position, we no longer have each Latin or Greek letter occurring exactly once in each of the diagonals. Nevertheless, when we translate these letter combinations to their numeric equivalents we will still have each number from 1 to 16 occurring exactly once, with no recurrences in the diagonals or anywhere else in the grid.

§18. Now in a process as described in §12, we deduce the values of each Greek and Latin letter, and from these we calculate the entries to go in each cell of the grid.

The conditions \(b + c = a + d\) and \(\alpha + \delta = \beta + \gamma\) come from the diagonals in the arrangement created in §17b.

\[b + c = a + d\] is 1 equation in 4 unknowns with the additional condition that \(a, b, c, \) and \(d\) take values from \(\{0, 4, 8, 12\}\). So choosing \(a\) determines \(d\), and choosing \(c\) determines \(b\).

Also, for the Greek letters, \(\alpha + \delta = \beta + \gamma\) is one equation with four unknowns taking values from \(\{1, 2, 3, 4\}\). So choosing \(\alpha\) determines \(\delta\) and choosing \(\beta\) determines \(\gamma\).

Bearing in mind that juxtaposition indicates addition, we can verify the square shown. For example, in the first column, \(b\delta = 4 + 8 = 8\); \(c\gamma = 8 + 3 = 11\); \(a\beta = 0 + 2 = 2\); \(d\alpha = 12 + 1 = 13\).

The sum for each row, column, and major diagonal is 34, as it ought to be according to the table in §2.
Indeed, a great many other forms are able to be made in this way, by transposing rows or columns.

§19. Neither is it absolutely required that each row or column have all the Latin and Greek letters occurring in it, as rows and columns can be made with only two Latin and Greek letters, providing that the sum of them is the same as the sum of all four letters.

It is indeed useful to construct figures with this special method, and with much work a particular rule can be made for placing each Latin letter with each Greek letter, such that while there is not a single sum for all the lines, each Latin letter is still combined with each Greek letter.

§20. To give an example of this method, first we set that

\[ a + d = b + c, \]

and we place the Latin letters as follows,

\[
\begin{array}{cccc}
  a & a & d & d \\
  d & d & a & a \\
  b & b & c & c \\
  c & c & d & d \\
\end{array}
\]

where clearly for all the lines the sum of the numbers is the same.

Notes:

§19 Now Euler considered Latin squares with the same letters occurring in a row or column, as some of the earlier squares we saw had recurrences in the diagonals. This does not violate the conventions of a Latin Square once we assign number values to the letters and make sure that each cell receives a unique number. This also pre-figures the use of Latin Squares in experimental design.

§20. Since \( a, b, c, \) and \( d \) are elements of \( \{0, 4, 8, 12\} \) it follows that

\[ a + b + c + d = 24 \] so we must have

\[ a + d = b + c = 12/ \]

An inspection of the bottom row reveals that the sum of that row is 30 while the sum of the other rows is 24. Moreover, the equation \( a+d = b+c \) suggests that the bottom row should be \( c \ c \ b \ b \) rather than \( c \ d \ d \ d \) as shown. This may be a typo by the translator, as the next figure with Greek and Latin characters shows the bottom row containing the Latin figures that one would expect:

\( c \ c \ b \ b \).
Then, the Greek letters are combined with each of their Latin equivalents in the left diagonal, since the two corresponding letters placed on either side of this line are different, and then the Greek letters are combined with their reflected equivalents, according to which the following figure is made:

\[
\begin{array}{cccc}
aα & aδ & dβ & dγ \\
daα & dδ & aβ & aγ \\
bδ & ba & cγ & cβ \\
cδ & ca & bγ & bβ \\
\end{array}
\]

where for the Greek letters it is necessary that it be taken:
\[
α + δ = β + γ.
\]
thus, if we take
\[
a = 0, b = 4, c = 8, d = 12 \text{ and } α = 1, β = 2, γ = 3 \text{ and } δ = 4,
\]
such a magic square is seen:

\[
\begin{array}{cccc}
1 & 4 & 14 & 15 \\
13 & 16 & 2 & 3 \\
8 & 5 & 11 & 10 \\
12 & 9 & 7 & 6 \\
\end{array}
\]

§21. In this way, many other figures can be formed, for instance the following:

\[
\begin{array}{cccc}
aα & dβ & aδ & dγ \\
bδ & cγ & ba & cβ \\
dα & aβ & dδ & ay \\
cδ & bγ & ca & bβ \\
\end{array}
\]

where it is clear for the Latin letters to take
\[
a + d = b + c,
\]
and for the Greek letters,
\[
α + δ = β + γ
\]
from which, if we took the values from above, the following magic square will be seen:

\[
\begin{array}{cccc}
1 & 14 & 4 & 15 \\
8 & 11 & 5 & 10 \\
13 & 2 & 16 & 3 \\
12 & 7 & 9 & 6 \\
\end{array}
\]

Notes:
Thus, we have on the diagonal aα in (1,1), dδ in (2,2), cγ in (3,3), and bδ in (4,4).

Then, treating this diagonal as an axis of symmetry, we reflect to obtain as follows: aδ in (1,2) and da in (2,1); dβ in ((1,3) and bδ in (3,1); dy in (1,4) and cδ in (4,1), etc.

By exploiting natural correspondences between Latin and Greek letters and the left diagonal as an axis of symmetry, Euler combined each Latin and Greek letter in an orderly manner such that each combination is expressed and none is repeated.

Then, by his mapping of these letters into values to be inserted in his statement of the remainder theorem, §3, a Latin square in which every number from 1 to \(x^2\) occurs exactly once, and each row, column, and major diagonal sum to \(x(1+x^2)/2\)

So in the first column of the figure shown, we have aα = 0 + 1 = 1; dα = 12 + 1 = 13; bδ = 4 + 4 = 8; \(cδ = 8 + 4 = 12\), etc.

§21 This figure was obtained by interchanging the two middle rows and also by interchanging the two middle columns of the previous figure.

These are the same relations as before.

Again we may verify that these numbers come from the letter combinations shown above. For example in the first column we have aα = 0 + 1 = 1; dδ = 4 + 4 = 8; dα = 12 + 1 = 13; and cδ = 8 + 4 = 12. As before, all rows, columns, and major diagonals sum to 34.
§22. In all of these forms, the sums of the Latin and Greek letters come to the same sum. Others can also be made, which do not follow any pattern, where the same sum for all the values is still obtained, but it would be futile to consider these anomalies, because chance plays such a great part with them that no fixed pattern can be given for them, and so in the following types, chance will be particularly kept in mind, so that the values of the Latin and Greek letters are not restricted.

III. Types of squares divided into 25 cells

§23. Therefore, for this type the five Latin letters a, b, c, d, e occur, and the five Greek letters α, β, γ, δ, ε for which the values of the former are 0, 5, 10, 15, 20, and the values of the latter 1, 2, 3, 4, 5; both of these letters must be inscribed in the cells of the square in an arrangement such that all the letters occur in each row, column and both diagonals.

§24. First, therefore, we inscribe all the Latin letters in order in the top row of the square, and then we fill up the left diagonal with letters such that the same letter does not occur twice in any of the remaining lines, with there being more than one way for this to be done. Once this line has been made, the other diagonal is immediately determined, and this next figure can be seen:

aε bδ cγ dβ eα
eβ cα dδ aγ be
da cγ bβ ce aδ
bγ dε aα eδ cβ
cδ aβ εε bα dγ

Notes

§23. By earlier convention, the Latin letters represent multiples of x, the base number, now 5; while Greek letters represent the numbers 1 through 5.

§24. The order of the Latin letters in the top row is completely arbitrary. Next, the left diagonal can also be filled in with some choices to be made. However, a bit of experimentation shows that, once these are assigned, the right diagonal is completely determined by the constraint that each letter should occur exactly once in each row and column. Here it is possible to have each letter occur exactly once in the right diagonal as well.
Then below the middle cell is written a and above it d, from which the middle column is completed, and then the remaining lines are determined immediately.

§25. For the Greek letters it is not helpful to use one of the diagonals, but if we consider the middle column, we find that there are different letters in the corresponding cells on both sides of it, so thus we write the Greek equivalents of the Latin letters in this column, and the Greek letters in the places of their reflected Latin equivalents, which is how we made this figure.

§26. Clearly no restrictions are prescribed for this figure, and in fact the Latin and Greek letters can take any number. Since with five letters there are 120 possible permutations, here altogether 14400 variations can be seen.

§27. If we permute the rows or columns between themselves, we can obtain many other forms, for which we must however set certain dependencies on the diagonals; for instance, if the first column is placed at the end, the following form is seen:

\[
\begin{array}{cccc}
  \text{b} & \text{c} & \text{d} & \text{e} \\
  \text{c} & \text{d} & \text{a} & \text{e} \\
  \text{e} & \text{b} & \text{c} & \text{d} \\
  \text{d} & \text{a} & \text{e} & \text{c} \\
  \text{a} & \text{e} & \text{b} & \text{d} \\
\end{array}
\]

In fact it was necessary to place a (4,3) and d in (2,3). An attempt to interchange these two made it necessary to place d in (5,3), a contradiction since d is already in row 5, in (5,5).

§25 The reader who is interested in constructing Sudoku grids “from scratch” that is, from a blank diagram, has now seen enough to know how to begin. Place the numbers 1 through x in the top row and in the axis of symmetry. From here it is possible to make one more choice, then the remaining cells will be determined by the rule that each number should occur once in each row and column. In some cases it is possible to have the each number occur once in the other diagonal as well.

§26. “Clearly no restrictions are prescribed for this figure…” it is not at all clear what is meant here. For the Latin letters must take values from \{0, 5, 10, 15, 20\} and the Greek letters must take values from \{1, 2, 3, 4, 5\}; moreover, each row and column must sum to the same number, \(5(1 + 25)/2 = 65\) as in §3. In fact in §27 we will see two equations in five variables which will show how these numbers are to be chosen, similar to the equations we saw in §21, §18, and §12.
where in fact in all the rows and columns all the letters occur; for the diagonals to be satisfied at the same time, this sum

\[ 3c + b + d + 3 \delta + \beta + \varepsilon \]

and this one

\[ 3a + b + c + 3 \varepsilon + \alpha + \beta \]

are set to be equal to the sum of all the Latin and Greek letters, namely

\[ a + b + c + d + e + \alpha + \beta + \gamma + \delta + \varepsilon, \]

and when collected, these two equations follow:

\[ 2c + 2\delta = a + e + \alpha + \gamma \]

and

\[ 2a + 2\varepsilon = d + e + \gamma + \delta, \]

whose conditions could be satisfied in many ways, where indeed the Latin and Greek letters can be determined such that

1) \( 2c = a + e, \)
2) \( 2a = d + e, \)
3) \( 2\delta = \alpha + \gamma, \) and
4) \( 2\varepsilon = \gamma + \delta. \)

Notes:

\[ \text{§27} \]

\[ 3c + b + d + 3 \delta + \beta + \varepsilon \]

is from the left diagonal

\[ 3a + b + c + 3 \varepsilon + \alpha + \beta \]

is from the right diagonal

Substituting, this is

\[ 1 + 2 + 3 + 4 + 5 + 0 + 5 + 10 + 15 + 20 \]

= 65, as shown above.

Equations 1), 2), 3), and 4) can be obtained from the previous equations by separating out the Latin and Greek letters. Equations 1 and 3 come from the first equation above, and equation 2 and 4 come from the second equation above.

1) \( 2c = a + e, \)
2) \( 2a = d + e, \)
3) \( 2\delta = \alpha + \gamma, \) and
4) \( 2\varepsilon = \gamma + \delta. \)
It is clear that the first two of these are satisfied if the letters d, b, a, c, e constitute an arithmetic progression, where it would be obtained that d = 0, b = 5, a = 10, c = 15 and e = 20; and the two remaining conditions are met if the Greek letters in the order α, β, δ, ε, γ proceed in arithmetic progression, with it obtained that α = 1, β = 2, δ = 3, ε = 4 and γ = 5, from which such a square will be seen:

\[
\begin{array}{cccc}
8 & 20 & 2 & 21 \\
16 & 3 & 15 & 9 \\
25 & 7 & 19 & 13 \\
4 & 11 & 23 & 17 \\
12 & 24 & 6 & 18 \\
\end{array}
\]

where clearly all sums are equal to 65.

§28. However, distributing the letters is by no means simple work, and requires careful consideration. In particular for the above types where many elements remain at our discretion, the number of such figures is very large; removing a restriction causes much work, because there is no clear restriction prescribed for the values of the letters. If the letter c takes the middle value, which is 10, with the others remaining at our discretion, we can fill one diagonal with the letter c, from which the other letters follow naturally, and such a figure can be seen:

In fact, it is reasonable to suppose that d, b, a, c, e are an arithmetic progression, since they are all multiples of 5.

§28.

That is, the possible values for the Latin numbers are 0, 5, 10, 15, 20 and as we saw earlier the middle cell must take the middle value when x is odd. Now, filling the left diagonal with c imposes some structure as we saw in §12 with a 3x3 square.
Now in the middle row, each of the Greek letters are written with their Latin equivalents, and then on either side of this the Greek equivalents are reflected, such that the following form is seen:

\[
\begin{array}{ccccc}
\alpha & \beta & \gamma & \delta & \epsilon \\
\beta & \epsilon & \alpha & \delta & \gamma \\
\gamma & \delta & \epsilon & \alpha & \beta \\
\delta & \gamma & \epsilon & \beta & \alpha \\
\epsilon & \delta & \beta & \alpha & \gamma \\
\end{array}
\]

from which clearly \( \gamma \) takes the middle value, which is 3; if we choose the following ordering:

\[
\begin{align*}
\alpha &= 0, & \beta &= 5, & \gamma &= 10, & \delta &= 15, & \epsilon &= 20 \\
\alpha &= 1, & \beta &= 2, & \gamma &= 3, & \delta &= 4, & \epsilon &= 5,
\end{align*}
\]

the following magic square will be seen:

\[
\begin{array}{ccccc}
14 & 20 & 21 & 2 & 8 \\
10 & 11 & 17 & 23 & 4 \\
1 & 7 & 13 & 19 & 25 \\
22 & 3 & 9 & 15 & 16 \\
18 & 24 & 5 & 6 & 12 \\
\end{array}
\]

§29. By the given method for forming odd squares by switching the values, this figure is formed:

\[
\begin{array}{ccccc}
11 & 24 & 7 & 20 & 3 \\
4 & 12 & 25 & 8 & 16 \\
17 & 5 & 13 & 21 & 9 \\
10 & 18 & 1 & 14 & 22 \\
23 & 6 & 19 & 2 & 15 \\
\end{array}
\]

Notes:

Here the left diagonal is filled with c, the top row is filled arbitrarily, and then the contents of most of the other cells are determined, as described in the notes on §25.

That is, the third row is written as \( a\alpha \ b\beta \ c\gamma \ d\delta \ e\epsilon \)

So, for example, \( \alpha \gamma \) appears in (2,2) while \( \alpha \gamma \) appears in (4,2), and so on, with symmetry about the third row.

Since the Greek letters are take values from \( \{1,2,3,4,5\} \) and the middle cell in an odd square takes the middle value, it follows that \( \gamma = 3 \) so \( c\gamma = 10 + 3 = 13 \).

As in §9. Note that this gives a non-trivial rearrangement, not simply a reflection in the center cell as we saw with the 3x3 square in §9.
which, with it restricted by our formulas, we first consider the left diagonal in for $c = 10$, and take
$\delta = 1$, $\alpha = 2$, $\gamma = 3$, $\epsilon = 4$ and $\beta = 5$, and then
$b = 0$, $d = 20$, $a = 15$, $e = 5$, where from these values this square is made.

§30. It is possible to discover other types than the forms that satisfy these rules, and it is indeed possible to thus greatly increase the number of magic squares. Yet it is hardly ever possible to be certain that we have exhausted all the possibilities, although the number of them is certainly not infinite. Certainly without doubt, such an investigation for finding a more general rule for using in different situations would still not work in many cases. However, it would still be very beautiful to add to the theory of combinations such a method.

IV. Types of squares divided into 36 cells

§31. Since the number of variations here is exceedingly large and there are many determinations remaining at our discretion, we produce a specific rule here, with which the Latin and Greek letters can easily be arranged, where the six Latin letters take such values:

$$a + f = b + e = c + d$$

and similarly for the Greek letters,

$$\alpha + \zeta = \beta + \epsilon = \gamma + \delta$$

Notes:

Compare this with
$a = 0$, $b = 5$, $c = 10$, $d = 15$, $c = 20$
and
$\alpha = 1$, $\beta = 2$, $\gamma = 3$, $\delta = 4$, $\epsilon = 5$ from §28
and we see that the equations from §27
1) $2c = a + e$,
2) $2a = d + e$,
3) $2\delta = \alpha + \gamma$, and
4) $2\epsilon = \gamma + \delta$.
are still satisfied. We have some leeway here because the Latin and Greek letters are only partially determined by the equations from §27; moreover, 5 may be represented by either a Latin or a Greek letter.

§31. $a + f = b + e = c + d$: and
$\alpha + \zeta = \beta + \epsilon = \gamma + \delta$

It seems that Euler picked these equalities from the symmetries of these letters when listed:
$a$ $b$ $c$ $d$ $c$ $f$ and
$\alpha$ $\beta$ $\gamma$ $\delta$ $\epsilon$ $\zeta$,
pairing the first letter with the last, the second with the second-last, and the two middle letters. Since this is their order in the alphabet, this would make it easier to permute them in a systematic way and make sure all combinations are included.
and by its similarity to §20, in each row, we inscribe two Latin letters, and then in each of the columns in the same way, we place two Greek letters, and 14 in this way the following figure is obtained:1

\[
\begin{array}{cccc}
\alpha & a & a & \alpha \\
\alpha & \alpha & a & f & f & f \\
\alpha & a & f & a & a & a \\
\end{array}
\]

§32. Thus it can be easily seen that the letters can be arranged in all the even types successfully, and for the odd types this can be done with the method described earlier, in which the letter that takes the middle value is repeated in one of the diagonals and in the other the letters are arranged appropriately. Therefore, for however many cells a given square has, it is always in our power to construct many magic squares, even if these rules that have been given are particular.

1Translator: As noted in the reprint of this paper in the Opera Omnia by the editor, this arrangement does not indeed make a magic square, as in one diagonal \( b\beta \) is placed twice, and in the other diagonal \( e\epsilon \) appears twice. However, the editor notes that in Euler’s Recerches sur une nouvelle espèce de quar’és magiques, Verhandelingen uitgegeven door het zeeuwisch Genootschap der Wetenschappen te Vlissingen 9 (1782), 85-239, reprinted in the same volume of the Opera Omnia as this paper, Euler does in fact give the following magic square with 36 cells:

\[
\begin{array}{cccccccc}
3 & 36 & 30 & 4 & 11 & 27 \\
22 & 13 & 35 & 12 & 14 & 15 \\
16 & 18 & 8 & 31 & 17 & 21 \\
28 & 20 & 6 & 29 & 19 & 9 \\
32 & 23 & 25 & 2 & 24 & 5 \\
10 & 1 & 7 & 33 & 26 & 34 \\
\end{array}
\]

Notes:

§31 Thus the first row is \( a \ a \ a \ f \ f \ f \) because \( a+f \) are paired in the first set of equations in §31. Then the next row is \( f \ f \ f \ a \ a \ a \). The next two rows are \( b \ b \ b \ e \ e \ e \) and \( e \ e \ e \ b \ b \ b \) because \( b+e \) are paired in the equations in §31. The final two rows are therefore \( c \ c \ c \ d \ d \ d \) and these same letters, reversed. So the Latin arrangement, before we insert the Greek letters, looks like this:

\[
\begin{array}{cccc}
a & a & a & f & f & f \\
f & f & f & a & a & a \\
b & b & b & e & e & e \\
e & e & e & b & b & b \\
c & c & c & d & d & d \\
d & d & d & c & c & c \\
\end{array}
\]

Next we insert the Greek letters in the columns by a similar rule, guided by the equations

\[
\alpha + \zeta = \beta + \varepsilon = \gamma + \delta
\]

So the first and second columns consists of three \( \alpha \)’s and three \( \zeta \)’s; the third and fourth columns consist of three \( \beta \)’s and three \( \varepsilon \)’s, and the fifth and sixth columns consist of three \( \gamma \)’s and three \( \delta \)’s.

§32. Euler may have becoming tired at this point, as he did not assign values for these letters and he left an error on the diagonals. Still, this work is highly commendable, for it shows how we may systematically create a Latin square with all entries unique, or, by using only the Latin letters, we may create a Sudoku with each entry occurring exactly once in each partition. This work is even more remarkable because Euler was old, and blind, when he wrote (actually dictated) this paper; thus, he did all of these calculations mentally. In a later paper, printed in 1782, he corrected this error, found values for the Latin and Greek letters, created a correct 6x6 Latin square, and showed how to generalize his method.