AN EMPIRICAL APPROACH TO THE ST. PETERSBURG PARADOX

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Probability is well known for its counter-intuitive ideas and results. During a recent History of Mathematics class, the first author of this note asked his students, including the second author, to read about one such idea commonly called the St. Petersburg Paradox. In this problem, a gambler is to play the following game against the house:

Game 1. A gambler flips a fair, 2-sided coin. If she flips “heads” on the first flip, the house pays her $2. If the first heads is the second flip, the house pays $4. In general, if heads first comes up on the nth flip, the house pays $2^n.

How much should the gambler pay to play this game? Or in modern terminology, what is the expected value of this game? The answer turns to be paradoxical, and is the reason the game is still discussed today.

A bit of history

The St. Petersburg paradox was invented by Nicolas Bernoulli, and first described in a letter to Montmort in 1713 (see [2]). By the standard calculation of expected value $E$, we multiply each possible payoff by the probability of winning that payoff, and add the products, finding that

\[
E = \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 4 + \frac{1}{8} \cdot 8 + \cdots + \frac{1}{2^n} \cdot 2^n + \cdots
\]

\[
= 1 + 1 + 1 + \cdots + 1 + \cdots
\]

\[
= \infty.
\]
Herein lies the paradox. Although there is no real scenario in which the house pays out infinitely many dollars, standard probability theory does give \( \infty \) as the expected value. Several 18th century mathematicians, including Nicholas and Daniel Bernoulli [1], the Compte de Buffon [4], and Leonhard Euler [7, 15], made attempts to resolve the paradox using utility functions to adjust the value of each additional dollar. John Maynard Keynes [11] and Paul Samuelson [14] are among more recent economists who have devoted considerable attention to its resolution. Thus, the St. Petersburg paradox has contributed to development of utility theory.

While utility theory can resolve the problem, several other methods have also been established. One calls for a cap on the amount the house can pay out, creating a finite sum. Similarly, some suggest setting a likelihood threshold such that if the odds of some event are less than a given value, the gambler assumes the event will never occur. This also creates a finite expectation by setting a maximum number of flips that can occur in a game. Still others have used “trimmed sums” to obtain a finite expectation. (See [14, 16, 5, 8] for modern treatments of these modified St. Petersburg games.)

One textbook we used in our class was Boyer and Merzbach [3], which describes an experiment by the 18th century naturalist the Compte de Buffon, who played the St. Petersburg game an astounding 2048 times. He found, in our notation, an average payout of $9.82. (It seems he did not do all the work himself, however: “I have played this game two thousand forty-eight times by having a coin thrown into the air by a child...”) The second author was intrigued by the experiment, and although Buffon was limited to using and flipping an actual coin, we have the luxury of living in the age of rapid electronic calculation. Before class was over, we wrote a program which simulated the St. Petersburg game. We played it a million times, and we found the average payoff to be about $18.32. This was so much greater than the value given in [3] that we decided to investigate further.
Initial Investigations

We simulated a St. Petersburg game played $n$ times. The average winnings over the $n$ games are reported in Table 1. These figures correspond closely to empirical results in the literature (see [10] for a typical example).

<table>
<thead>
<tr>
<th>$n$</th>
<th>Avg. Winnings (dollars)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>7.80</td>
</tr>
<tr>
<td><strong>2048</strong></td>
<td><strong>18.70</strong></td>
</tr>
<tr>
<td>5000</td>
<td>15.49</td>
</tr>
<tr>
<td>10000</td>
<td>14.93</td>
</tr>
<tr>
<td>50000</td>
<td>15.60</td>
</tr>
<tr>
<td>100000</td>
<td>21.15</td>
</tr>
<tr>
<td>500000</td>
<td>19.46</td>
</tr>
<tr>
<td>1000000</td>
<td>25.69</td>
</tr>
</tbody>
</table>

Table 1 is fairly interesting. It seems that the average per-game winnings depends rather strongly on the number of games played. This, however, turned out to be well known; in fact, after playing $n$ games, the most likely value of average winnings per game is $\log_2(n)$. What we were actually interested in was something more complex; we wanted not the most likely average payoff per game, but the distribution of winnings that come from doing what Buffon did – namely, playing 2048 St. Petersburg games. Specifically, we wondered whether the value he found is a representative value for playing the game 2048 times.

The experiment

With an eye toward understanding Buffon’s results, we defined a Buffon experiment to be playing the St. Petersburg game 2048 times. We took the result of a Buffon experiment, then, to be the average winnings per game over the 2048 total games. We simulated playing a million Buffon experiments; the results are in Figure 1.
We were taken aback at the resulting bizarre distribution. When we showed the graph around, several colleagues expressed surprise. One questioned our random number generator! What sense can we make of this picture? We immediately see that the most likely values are indeed those near $\log_2(2048) = 11$, as expected. However, the rest of the graph is surprising. Its comb-like, fractaline quality demands explanation.

In general, the average payoff in one Buffon experiment is largely determined by the largest payoff of any single game in the experiment. Notice, for example, the spike in the graph near $n = 42$. If even one game involves flipping heads sixteen consecutive times, the payoff for that game will be $2^{16} = 65,536$, and the average payoff for each of the 2048 games in that Buffon experiment will be at least $(65536/2048) = 32$. The other 2047 games will have about the same average as any given event, and will thus add about $32$ to the total. Thus we see a small spike for values about 11 greater than 64, 128, and 256, together with smaller spikes about 11 greater than 128+64, 256+64, etc.
A new experiment

The surprising nature of Figure 2 led us to try another approach. Consider that in any given event of Buffon experiment of 2048 games, the expected number of games that begin with precisely 11 tails is $2048 \cdot 1/(2^{11}) = 2048/2048 = 1$. Thus over the million Buffon experiments that we ran, there should be about one million individual games in which $2048$ is paid out. Similarly, we expect about 500,000 games in which $4096 = 2^{12}$ is paid, etc, and for any integer $n$, we expect about $1,000,000/2^{n-11}$ games which pay $2^n$ dollars.

Therefore we devised a new experiment. We created 1,000,000 bins representing the million Buffon experiments that we ran, and for each positive integer $n \leq 31 \approx \log_2(2048 \cdot 1,000,000)$, we randomly chose $1,000,000/2^{n-11}$ bins (rounded to the nearest integer) into each of which we placed a payoff of $2^n$ dollars. For any $n$, we allowed the same bin to be chosen more than once (after all, within one Buffon experiment, we could have more than one game with 17 consecutive tails). Each of these bins, therefore, ‘represents a mock Buffon experiment’. We then calculated the average payoff in each bin, plotted the distribution of the log of these averages, and found a graph of the same form and shape as Figure 1. Thus Figure 1 can also be thought of as the result of a Monte Carlo process of distributing powers of two representing possible winnings into bins.

Conclusion

As an answer our original question, we return to Buffon’s results. Recall that he gave $9.82$ as a good approximation to the expected value of the St. Petersburg game. Our data show that only 9.06% of Buffon experiments have an average payout of less than $10$. The mean “average payout” was $19.60$, and the median was $12.63$. Therefore Buffon’s empirical value, the one often reported in histories of the subject, is, in fact, fairly unlikely. (Indeed, we now wonder whether Buffon or his accomplice cheated. Perhaps he played a single game of large payout, but ignored it as unlikely and unrepresentative of what “should” happen.)
It has become a truism in probability and statistics that in order to understand a variable, it is not enough to know its mean – one must also know the distribution. Our approach, looking at the distribution of average payoffs obtained by playing a fixed number of St. Petersburg games, sheds new insight on the possible results of this centuries-old problem, and also generates a beautiful, surprising, and seemingly new distribution.

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Summary

The St. Petersburg game is a probabilistic thought experiment. It describes a game which seems to have infinite expected value, but which no reasonable person could be expected to pay much to play. Previous empirical work has centered around trying to find the most likely payoff that would result from playing the game $n$ times. In this paper, we extend this work to the distribution of all possible values which could result from this experiment. We use this distribution – with a surprising fractal-like pattern – to examine the unlikely nature of the most famous experiment on this game, the results of the Compte de Buffon’s playing the game 2048 times.

References


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