Advanced Analysis III | Spring 2008 | Final Exam

Due Friday May 30 by 3 pm

Instructions. Please do not discuss these problems with any classmates or other faculty. You can ask me questions if you are really stuck.

1. Let (X, ρ) be a complete metric space and $f : X \to X$ a function.

   (a) If f^n is a contraction for some positive integer n, prove that f has a unique fixed point. The notation being used here is $f^2 = f \circ f$, $f^3 = f \circ f \circ f$, etc.

   (b) Find an example which shows that the condition $\rho(f(x), f(y)) < \rho(x, y)$ is not sufficient for the existence of a fixed point. I would suggest trying to find an example where $X = \mathbb{R}$ or $X = (0, \infty)$. Don't try with X compact.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous. Define $f_j(x) = f(x + \frac{1}{j})$ for $j \in \mathbb{N}$. Prove that the sequence $\{f_j\}$ converges uniformly to f.

3. Let $f \in C([a, b])$ with $f(x) > 0$ for all $x \in [a, b]$. Prove that the function $\frac{1}{f}$ is bounded on $[a, b]$.

4. Consider the initial value problem $\frac{dy}{dx} = x + y, \qquad y(0) = 0$. Use Picard iteration (like we did in class) to find the solution.