# EIGENVALUES IN FILLED JULIA SETS

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The study of Julia sets, the Mandelbrot set, and complex dynamics in general is an exciting and active area of research that students find fascinating. Exposing our students to these contemporary topics as early as possible is a worthwhile endeavor. Many wonderful materials have already been developed along these lines. An incomplete list of resources, but among my favorites, are [4], [5], and [9]. In this article we see how Julia sets can be introduced very naturally in a junior level linear algebra course. We do not discuss the complicated (actually chaotic) dynamics or the fractal geometry involving Julia sets, but limit ourselves to using standard topics from linear algebra to prove that Julia sets are lurking in the background disguised as eigenvalues. In fact, with the exception of the last theorem (whose proof uses Cauchy's integral formula), our discussion does not rely upon any mathematics beyond the topics typically covered in a junior level linear algebra course.

## What are Julia sets?

The study of the dynamics of complex analytic functions began during the First World War. At this time, Gaston Julia (1893-1978) and Pierre Fatou (1878-1929) independently published many remarkable results that laid the groundwork for the field. Their results were mostly forgotten until Benoit Mandelbrot in the late 1970's produced the first computer images that have now become so popular. Julia sets and Fatou sets are named in their honor.

Let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial and define $p^n(z) = (\underbrace{p \circ \cdots \circ p}_{n \text{ times}})(z)$. The *basin (of*

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*attraction)* of $\infty$ is defined as

$$A(\infty) = \{z \in \mathbb{C} \ : \ |p^n(z)| \to \infty\}.$$

The *Julia set*, denoted by $J$, is defined as the boundary of the basin of $\infty$ and the *filled Julia set of* $p$ is defined as

$$K = \{z \in \mathbb{C} \ : \ |p^n(z)| \not\to \infty\} = \mathbb{C} \setminus A(\infty).$$

Although our discussion focuses on the filled Julia set, we should mention that the *Fatou set* is defined as the complement of the Julia set.

**Example 1.** *Let* $p(z) = z^2$*. Then for* $z = re^{i\theta}$, $p^n(z) = r^{2^n} e^{2n\theta i}$ *and* $|p^n(z)| = r^{2^n}$*. Therefore,* $|p^n(z)| \to \infty$ *if and only if* $|z| = r > 1$*. It follows that*

$$A(\infty) = \{z \in \mathbb{C} \ : \ |z| > 1\}, \ J = \{z \in \mathbb{C} \ : \ |z| = 1\}, \ \text{and } K = \{z \in \mathbb{C} \ : \ |z| \leq 1\}.$$

*A similar argument as above can be used to show that the Julia set for* $p(z) = z^m$ *with* $m \geq 2$ *is the unit circle.*

Most Julia sets are more interesting than a circle and analytically describing the set as we did in the previous example is often too difficult However, several algorithms for visualizing Julia sets have become very popular (See [4] and [9]). We note that the images in this article were produced using the algorithm found on page 233 of [4]. FIGURES 1-4 show the filled Julia set for selected quadratic functions of the form $p(z) = z^2 + c$ with $c \in \mathbb{C}$. In FIGURES 3 and 4 the Julia set equals the filled Julia set (the filled Julia set has empty *interior*) and is called a *dendrite* and a *Cantor set*, respectively.

**Figure 1** Filled Julia set for $p(z) = z^2 + .5i$

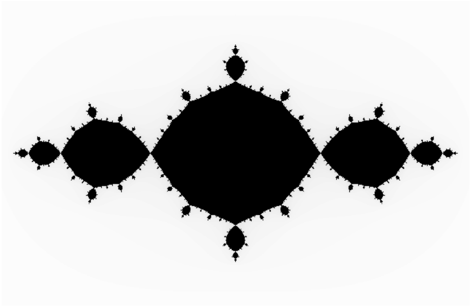

**Figure 2** Filled Julia set for $p(z) = z^2 - 1$

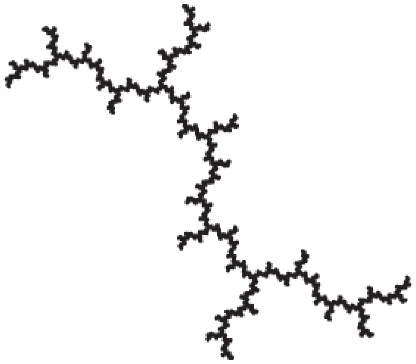

**Figure 3** Julia set for $p(z) = z^2 + i$

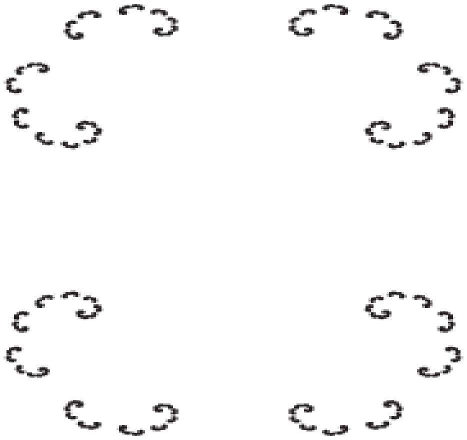

**Figure 4** Julia set for $p(z) = z^2 + .5$

There is of course much more that could be said (and is fun to learn) about Julia sets. For now we remain content to see how Julia sets appear in a linear algebra setting.

## Linear algebra basics

Before beginning our hunt for Julia sets we need to collect some necessary tools. Let $\mathbb{C}^{k\times k}$ be the space of all $k\times k$ matrices over $\mathbb{C}$. For $Z\in\mathbb{C}^{k\times k}$, we define the *spectrum* of $Z$ by

$$\sigma(Z)=\{\lambda\in\mathbb{C}:\lambda\text{ is an eigenvalue of }Z\}.$$

The *spectral radius* of $Z$ is defined by

$$\rho(Z)=\max\{|\lambda|:\lambda\in\sigma(Z)\}.$$

For a matrix norm on $\mathbb{C}^{k\times k}$ we use the *Frobenius norm* defined by

$$\|Z\|=\left(\sum_{j=1}^{k}\sum_{i=1}^{k}|z_{i,j}|^2\right)^{1/2}.$$

Besides the properties satisfied by general norms, the Frobenius norm (and any other matrix norm) satisfies the submultiplicative property

$$\|ZW\|\le\|Z\|\,\|W\|\ \text{ for all } Z,W\in\mathbb{C}^{k\times k}. \tag{1}$$

For a good introduction to matrix norms the reader is sent to [8].

Note if $\lambda$ is an eigenvalue for $Z$ with corresponding eigenvector $\mathbf{x}$ then, with $X\in\mathbb{C}^{k\times k}$ having all columns equal to $\mathbf{x}$, we have $ZX{=}\lambda X$ and

$$|\lambda|\,\|X\|=\|\lambda X\|=\|ZX\|\le\|Z\|\,\|X\|\,.$$

Dividing through by $\|X\|$ (how do we know $\|X\|\neq 0$?) we have $|\lambda|\le\|Z\|$ and get the following nice bound on the spectral radius.

$$\rho(Z)\le\|Z\|\,. \tag{2}$$

## Julia sets are eigenvalues

We are now ready to find out where Julia sets are hiding. Suppose $p(z)=a_mz^m+a_{m-1}z^{m-1}+\cdots+a_1z+a_0$ is a polynomial with complex coefficients. If $Z\in\mathbb{C}^{k\times k}$ then it makes sense to define

$$p(Z)=a_mZ^m+a_{m-1}Z^{m-1}+\cdots+a_1Z+a_0I$$

where $I$ is the $k \times k$ identity matrix. It is also very natural to consider the set

$$\mathcal{K} = \{Z \in \mathbb{C}^{k\times k} : \|p^n(Z)\| \not\to \infty\}$$

which is analogous to the filled Julia set of $p$. In fact, if $k = 1$ then $\mathcal{K} = K$. Hence, $\mathcal{K}$ generalizes the filled Julia set from $\mathbb{C}$ to $\mathbb{C}^{k\times k}$.

**Theorem 1.** *If $Z \in \mathcal{K}$ then $\sigma(Z) \subseteq K$. That is, if $Z$ is bounded under iteration by $p$, then the eigenvalues of $Z$ are contained in the filled Julia set of $p$.*

*Proof.* Suppose $Z \in \mathcal{K}$ and $\lambda \in \sigma(Z)$. A well known result called the *spectral mapping theorem* (which is not hard to prove) is that $p^n(\lambda) \in \sigma(p^n(Z))$. Using (2) we have

$$|p^n(\lambda)| \leq \rho(p^n(Z)) \leq \|p^n(Z)\| \not\to \infty.$$

It follows that $\lambda \in K$. Therefore $\sigma(Z) \subseteq K$. □

The converse of Theorem 1 is not true as the following example shows.

**Example 2.** *Let $p(z) = z^2$ and*

$$Z = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

*Clearly (why is this clear?), $-1$ is a repeated eigenvalue for $Z$. Since $p^n(-1) = 1$ for all $n \geq 1$, we conclude $\sigma(Z) \subseteq K$. However, direct computation shows*

$$p^n(Z) = \begin{bmatrix} 1 & -2^n \\ 0 & 1 \end{bmatrix}$$

*and $\|p^n(Z)\| \geq 2^n \to \infty$. Therefore, $Z \notin \mathcal{K}$.*

In the prior example note the eigenvalues for $Z$ lie on the boundary of $K$ (which is equal to the Julia set $J$) and $Z$ is not diagonalizable. However, if $Z$ is diagonalizable then $\sigma(Z) \subseteq K$ does imply that $Z \in \mathcal{K}$ regardless of whether or not any of the eigenvalues lie on the boundary of $K$. The argument is as follows. Suppose that $Z \in \mathbb{C}^{k\times k}$ is diagonalizable and $\lambda_1, \lambda_2, \cdots, \lambda_k$ are the eigenvalues of $Z$ (repetitions allowed). Then there exists a nonsingular matrix $S \in \mathbb{C}^{k\times k}$ such that $Z = SDS^{-1}$ where

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & & 0 \\ 0 & \lambda_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & & \\ & & & \ddots & 0 \\ 0 & \cdots & & 0 & \lambda_k \end{bmatrix} = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_k).$$

In addition, $p^n(Z) = SD^nS^{-1}$ where $D^n = \operatorname{diag}(p^n(\lambda_1), p^n(\lambda_2), \cdots, p^n(\lambda_k))$. Now

$$\|D^n\| = \left(\sum_{i=1}^{k} |p^n(\lambda_i)|^2\right)^{1/2} \leq \sqrt{k} \cdot \rho(p^n(Z))$$

and by (1)

$$\|p^n(Z)\| \leq \|S\| \, \|D^n\| \, \left\|S^{-1}\right\| \leq \|S\| \, \left\|S^{-1}\right\| \sqrt{k} \cdot \rho(p^n(Z)). \tag{3}$$

Combining (2) and (3) we obtain

$$\rho(p^n(Z)) \leq \|p^n(Z)\| \leq \|S\| \, \left\|S^{-1}\right\| \sqrt{k} \cdot \rho(p^n(Z)).$$

Thus, $\|p^n(Z)\| \not\to \infty$ if and only if $\rho(p^n(Z)) \not\to \infty$. We now have a partial converse to Theorem 1.

**Theorem 2.** *Suppose $Z$ is diagonalizable. If $\sigma(Z) \subseteq K$ then $Z \in \mathcal{K}$. That is, if the eigenvalues of $Z$ are contained in the filled Julia set of $p$, then $Z$ is bounded under iteration by $p$.*

**Example 3.** *Let $p(z) = z^3 - .5 - .1i$ and*

$$Z = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 1-i \\ 0 & 0 & -i \end{bmatrix}.$$

*The eigenvalues of $Z$ are $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = i$ with corresponding eigenvectors $\mathbf{x}_1 = [1,0,0]^T, \mathbf{x}_2 = [-1,1,0]^T$, and $\mathbf{x}_3 = [0,-1,1]^T$. It follows that $Z$ is similar to the diagonal matrix*

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -i \end{bmatrix}.$$

*It is easy to show that the eigenvalues of* $Z$ *are contained in the filled Julia set of* $p$*. Thus,* $Z$ *is bounded under iteration by* $p$*. The filled Julia set of* $p$ *along with the eigenvalues of* $Z$ *(highlighted) are shown in FIGURE 7. It is interesting to note that the filled Julia set of* $p$ *is not convex. For example, both* $0$ *and* $i$ *are in the filled Julia set but* $-.6i$*, which is on the line segment connecting* $0$ *and* $i$*, is no longer in the filled Julia set. In fact, if we replace the diagonal entry* $-i$ *with* $-.6i$ *in the matrix* $Z$*, then* $Z$ *is still diagonalizable,*$-.6i$ *is not in the filled Julia set of* $p$*, and* $Z$ *is no longer bounded under iteration by* $p$*. The number* $-.6i$ *is also highlighted in FIGURE 7.*

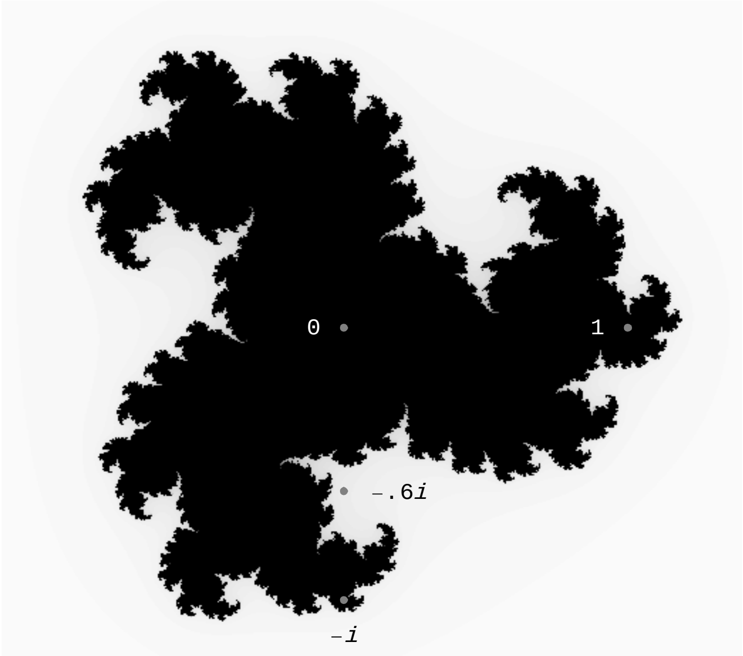

***Figure 7*** *The filled Julia set for* $p(z) = z^3 - .5 - .1i$*.*

The situation is more complicated if $Z \in \mathbb{C}^{k\times k}$ is not diagonalizable. As we saw in EXAMPLE 2, it is possible for $\sigma(Z) \subseteq K$ and $Z \notin \mathcal{K}$. This will not happen however if the eigenvalues of $Z$ are contained in the *interior* of $K$ (the *interior* of $K$ is denoted by $K^\circ$). Proof of this result requires *Cauchy's integral formula* for derivatives. But first we consider the *Jordan canonical form* of $Z$. Suppose $S^{-1}ZS =$diag$(J_1, J_2, \cdots, J_r)$ is the *Jordan canonical form* of $Z$ with

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \cdots & 0 \\ 0 & \lambda_i & 1 & & \vdots \\ & \ddots & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & & 0 & \lambda_i \end{bmatrix}$$

being an $m_i \times m_i$ *Jordan block where* $m_1 + \cdots + m_r = k$. Note that if $Z$ is diagonalizable then each Jordan block is $1 \times 1$.

As is shown in [6, Theorem 11.1.1]

$$p^n(Z) = S\mathrm{diag}(p^n(J_1), p^n(J_2), \cdots, p^n(J_r))S^{-1} \tag{4}$$

where

$$p^n(J_i) = \begin{bmatrix} p^n(\lambda_i) & (p^n)^{(1)}(\lambda_i) & & \cdots & \frac{(p^n)^{(m_i-1)}(\lambda_i)}{(m_i-1)!} \\ 0 & \ddots & \ddots & & \vdots \\ & \ddots & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & (p^n)^{(1)}(\lambda_i) \\ 0 & \cdots & & 0 & p^n(\lambda_i) \end{bmatrix}. \tag{5}$$

We can see from (4) that $Z \in \mathcal{K}$ if and only if $p^n(J_i) \in \mathcal{K}$ (Technically, we should write $Z \in \mathcal{K}$ and $p^n(J_i) \in \mathcal{K}_i$ as $\mathcal{K}$ contains $k \times k$ matrices and $\mathcal{K}_i$ contains $m_i \times m_i$ matrices). From (5) it is clear that in order to determine if $p^n(J_i) \in \mathcal{K}$, we must not only determine if the sequence $\{p^n(\lambda_i)\}$ is bounded as $n \to \infty$, but also if each of the sequences

$$\left\{(p^n)^{(1)}(\lambda_i)\right\}, \cdots, \left\{\frac{(p^n)^{(m_i-1)}(\lambda_i)}{(m_i-1)!}\right\}$$

is bounded as $n \to \infty$. One approach would be to consider the more general (and traditional) definition of Julia sets using *normal families*. However, this would lead us into a more advanced discussion involving *compact sets* and the *Arzelà-Ascoli theorem* (See [3]). We instead use the concept of a *locally bounded family*. The family of iterates $\{p^n\}$ is said to be a *locally bounded family* at $w$ if there exist $\delta > 0$ and $M > 0$ such that $|p^n(z)| \leq M$ for all $z$ with $|z - w| < \delta$ and for all $n \geq 0$. We require the following result which is stated without proof (See [3, §III.4])

**Proposition 1.** *If* $w \in K^\circ$ *then* $\{p^n\}$ *is a locally bounded family at* $w$.

We are now ready to prove our final result, which can be considered a generalization of Theorem 2 and a weak converse of Theorem 1.

**Theorem 3.** *If* $\sigma(Z) \subseteq K^{\circ}$ *then* $Z \in \mathcal{K}$. *That is, if the eigenvalues of* $Z$ *are contained in the interior of the filled Julia set, then* $Z$ *is bounded under iteration by* $p$.

*Proof.* Suppose $\sigma(Z) \subseteq K^{\circ}$. It suffices to show $p^n(J_i) \in \mathcal{K}$, where $J_i$ is a corresponding Jordan block with eigenvalue $\lambda_i$. Since $\lambda_i \in \sigma(Z)$, by Proposition 1 $\{p^n\}$ is a locally bounded family at $\lambda_i$. Therefore, there exist $\delta > 0$ and $M > 0$ such that $|p^n(z)| \leq M$ for all $z$ with $|z - \lambda_i| < \delta$. Recall the Cauchy integral formula for the derivative as applied to $p^n$,

$$(p^n)^{(1)}(w) = \frac{1}{2\pi i}\int\limits_C \frac{p^n(\xi)d\xi}{(\xi - w)^2}$$

where $C = \{z : |z - \lambda_i| = r\}$ with $r < \delta$. Hence, for all $w \in \{z : |z - \lambda_i| \leq \frac{r}{2}\}$,

$$\left|(p^n)^{(1)}(w)\right| \leq \frac{1}{2\pi}\int\limits_C \frac{|p^n(\xi)|\, d\xi}{|(\xi - w)^2|} \leq \frac{1}{2\pi}\int\limits_C \frac{4M d\xi}{r^2} \leq \frac{4M}{r}.$$

Therefore, $\left\{(p^n)^{(1)}\right\}$ is a locally bounded family at $\lambda_i$. In particular, the sequence $\left\{(p^n)^{(1)}(\lambda_i)\right\}$ is bounded. An induction argument can be used to conclude that for all $j \geq 1$ the sequence $\left\{(p^n)^{(j)}(\lambda_i)\right\}$ is bounded. It follows that $\|p^n(J_i)\| \not\to \infty$. $\square$

It is the dynamical properties of the Julia set that require us to limit the eigenvalues to belonging to the interior of the filled Julia set in the nondiagonalizable case. In particular, if the eigenvalues are in the interior of the filled Julia set then their behavior under iteration is much more predictable then if they were in the Julia set (i.e. the boundary). There are many more comparisons between eigenvalues, Julia sets, and dynamical systems in general that could be investigated. The reader is invited to explore these topics further.

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