Properties of Solutions of Autonomous Differential Equations

by

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A Basic Fact About Functions

Consider a function f defined on an open interval (a,b) such that f has a continuous derivative $f\,'$ on (a,b) which is always positive. Thus f is increasing on (a,b) and the range of f is the open interval (c,d), where

$$c = \lim_{x\to c^+} f(x),\ d = \lim_{x\to d^-} f(x)\,.$$

(Note that one or both of these limiting endpoint values may be infinite.) There is then an inverse function f^{-1} defined on (c,d) and it is also increasing and has a continuous derivative given by

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} \text{ for all } y \in (c,d)\,.$$

Example 1: Consider $f(x) = \sqrt{1+x}$, defined on $(-1,8)$. The above conditions are met and we have $(c,d) = (0,3)$. The inverse function, defined on $(0,3)$, is found by solving the equation $y = f(x)$ for x in terms of y and setting $f^{-1}(y) = x$. Thus, in the present case $y = \sqrt{1+x}$ has solution $x = y^2 - 1$, so $f^{-1}(y) = y^2 - 1$.

Example 2: Consider $f(x) = \ln(\frac{x}{1+x})$, defined on $(0,\infty)$. The above conditions are met and we have $(c,d) = (-\infty,0)$. The inverse function, defined on $(-\infty,0)$, is found by solving the equation $y = f(x)$ for x in terms of y and setting $f^{-1}(y) = x$. Thus, in this case we get $f^{-1}(y) = \dfrac{e^y}{1-e^y}$ for $y < 0$.

Solutions Approaching a Critical Point

Suppose $f(y)$ and $f\,'(y)$ are continuous on an interval $[a,b]$ and that

$$f(y) > 0 \quad \text{for} \quad y \in (a,b)$$
$$f(b) = 0$$

Without further analysis, we can say that any solution of $y' = f(y)$ such that $y(t) \in (a,b)$ has a positive derivative and hence $y(t)$ moves toward *b* as *t* increases. But actually, much more can be said.

Remark: We shall work with the assumption $f(y) > 0$. There are analogous results for cases where $f(y) < 0$.

Theorem 1: Given $y_0 \in (a,b)$, there is a solution of

$$y'(t) = f(y(t))$$
$$y(t_0) = y_0$$

that is defined (at least) for $t_0 \le t < \infty$ and satisfies $\lim_{t\to\infty} y(t) = b$.

Proof: To show this, define

$$G(y) = t_0 + \int_{y_0}^{y} \frac{1}{f(z)} dz \quad \text{for} \quad y \in (a,b).$$

Then *G* maps (a,b) into $\mathbb{R}$ and

$$G'(y) = \frac{1}{f(y)} > 0 \quad \text{for} \quad y \in (a,b)$$
$$G(y_0) = t_0$$

Therefore, G is an increasing function with domain the interval (a,b) and range some interval (c,d). Therefore, there is an inverse function $y(t)$ having domain (c,d) and range (a,b) and we have

$$y(t_0) = y_0$$
$$G(y(t)) = t \quad \text{for} \quad t \in (c,d)$$

Therefore,

$$G'(y(t))y'(t) = 1 \quad \text{for} \quad t \in (a,b)$$

and since

$$G'(y(t)) = \frac{1}{f(y(t))}$$

we see that

$$y'(t) = f(y(t)) \quad \text{for} \quad t \in (c,d)$$
$$y(t_0) = y_0$$

Next we shall argue that we must have $d = \infty$. If so, then since $y(t)$ is an increasing function with domain $(c,d) = (c,\infty)$ and range (a,b), we must have

$$\lim_{t\to\infty} y(t) = b$$

as asserted. Since we know

$$\lim_{y\to b^-} G(y) = d$$

we need only show

$$\lim_{y\to b^-} \int_{y_0}^{y} \frac{1}{f(z)}\,dz = \infty .$$

Recall that this limit is, by definition, the value of the improper integral

$$\int_{y_0}^{b} \frac{1}{f(z)}\,dz .$$

This is an improper integral because the integrand has a zero in the denominator at $z = b$ (recall that we assumed $f(b) = 0$).

Notice that just because we have an improper integral where the integrand has a zero in the denominator at one endpoint, it does *not* follow that the improper integral must diverge to infinity. For example, consider

$$\int_0^1 \frac{1}{\sqrt{1-x}}\,dx = \lim_{b\to 1^-} \int_0^b \frac{1}{\sqrt{1-x}}\,dx = \lim_{b\to 1^-} \left(-2\sqrt{1-x}\Big|_1^b \right) = \lim_{b\to 1^-} \left(-2\sqrt{1-b} + 2 \right) = 2 .$$

So why does the integral that we are interested in diverge to infinity? The answer lies in the assumption we made back at the beginning of this discussion, that $f(y)$ and $f'(y)$ are continuous on the interval $[a,b]$. Note this is a closed interval including the endpoint *b*. In the example above, $f(x) = \sqrt{1-x}$ *is* continuous on the closed interval $[0,1]$ but the derivative $f'(x) = \dfrac{-1}{2\sqrt{1-x}}$ is *not* continuous on the entire interval $[0,1]$, since it is undefined at $x = 1$. We digress from the proof of **Theorem 1** for a moment to prove the following Lemma.

Lemma 1: If $f(y)$ and $f'(y)$ are continuous on the interval $[a,b]$ with $f(y) > 0$ for $y \in (a,b)$ and $f(b) = 0$, then there is a function $h(y)$, continuous on $[a,b]$, such that

$$f(y) = h(y)(b-y) \quad \text{for} \quad y \in [a,b].$$

Proof: Let us define $h(y)$ as follows:

$$h(y) = \frac{f(y)}{b-y} \quad if \quad a \le y < b$$
$$h(b) = -f\,'(b)$$

Since $f(y)$ is continuous on $[a,b]$ it is clear that $h(y)$ is continuous on $[a,b)$. But $h(y)$ is also continuous at $y=b$ because

$$\lim_{y\to b} h(y) = \lim_{y\to b}\frac{f(y)}{b-y} = \lim_{y\to b}\frac{f(y)-f(b)}{b-y} = -\lim_{y\to b}\frac{f(y)-f(b)}{y-b} = -f\,'(b),$$

where we have used the assumption that $f(b)=0$. This proves **Lemma 1**. ■

Now we return to the proof of **Theorem 1**. The function $h(y)$ given to us by Lemma 1 is continuous and non-negative on the closed interval $[a,b]$ so it has a maximum value on that interval, that is, there is a constant $M>0$ such that

$$0 \le h(y) \le M \quad \text{for} \quad a \le y \le b.$$

Then,

$$\int_{y_0}^{y}\frac{1}{f(z)}dz = \int_{y_0}^{y}\frac{1}{h(z)(b-z)}dz \ge \int_{y_0}^{y}\frac{1}{M(b-z)}dz = \frac{-1}{M}\ln(b-z)\Big|_{y_0}^{y} = \frac{1}{M}\ln\left(\frac{b-y_0}{b-y}\right)$$

.

Therefore, taking the limit as $y \to b^-$, we see

$$\lim_{y\to b^-}\int_{y_0}^{y}\frac{1}{f(z)}dz \ge \lim_{y\to b^-}\frac{1}{M}\ln\left(\frac{b-y_0}{b-y}\right) = \infty,$$

showing that our improper integral diverges, as asserted. ■

Solutions Approaching Infinity

Now suppose that $f(y)$ and $f'(y)$ are continuous on an interval (a,∞) and that $f(y) > 0 \quad for \quad a < y < \infty$. If $y_0 \in (a,\infty)$ then any solution of the initial-value problem

$$y'(t) = f(y(t))$$
$$y(t_0) = y_0$$

must be an increasing function of time. We will actually show that $y(t) \to \infty$ as t approaches the right endpoint of the domain of $y(t)$. Here are two examples to illustrate what is being claimed here.

Example 1: Consider

$$y' = y^2$$
$$y(0) = 1$$

Take the interval $(a,\infty) = (0,\infty)$. The solution of this IVP is

$$y(t) = \frac{1}{1-t}.$$

Note that the largest interval starting at $t = 0$ on which $y(t)$ is defined is $(-\infty,1)$. We see that

$$\lim_{t\to 1^-} y(t) = \infty,$$

as asserted. Note that here the domain of the solution of the IVP is $(-\infty,1)$, but we see that, indeed, as t approaches the right end of this interval the solution goes to ∞.

Example 2: Consider

$$y' = 2y$$
$$y(0) = 1$$

Here, the basic conditions are met on the interval $(0,\infty)$. The solution of the IVP is

$$y(t) = e^{2t} \quad for \quad -\infty < t < \infty$$

and we see, as asserted, that

$$\lim_{t\to\infty} y(t) = \infty.$$

So, in this case, the solution is defined on $(-\infty,\infty)$ and as t approaches the right end of this interval the solution goes to ∞.

We now state our result.

Theorem 2: Assume that $f(y)$ and $f'(y)$ are continuous on an interval (a,∞) and that $f(y)>0 \quad for \quad a<y<\infty$. If $y_0 \in (a,\infty)$ then the solution of the initial-value problem

$$y'(t) = f(y(t))$$
$$y(t_0) = y_0$$

is defined on an interval (α,β) containing t_0 such that

$$\lim_{t\to\beta^-} y(t) = \infty$$

Proof: We begin as before by defining

$$G(y) = t_0 + \int_{y_0}^{y} \frac{1}{f(z)} dz \quad \text{for} \quad y \in (a,\infty).$$

This is defined for all $y \in (a,\infty)$, since $f(z)>0$ for all z in that interval and, as before, we note $G(y_0)=t_0$. We see $G(y)$ is an increasing function of y on (a,∞) and therefore the range of G is an interval (α,β), where β may be ∞ or a finite value. We let $y(t)$ be the inverse function of G and note that the domain of $y(t)$ is (α,β) and the range is (a,∞). It then follows that

$$\lim_{t\to\beta^-} y(t) = \infty,$$

as asserted. $\blacksquare$