Notes on the Discrete Fourier Transform

by

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Introduction

Suppose we have a “signal” \( x(t) \) that is periodic of period \( L \). We divide \([0,L]\) into \( N \) equal parts and “sample” the signal at the evenly spaced times

\[
t_n = \frac{nL}{N}, \quad 0 \leq n \leq N - 1
\]

obtaining the discrete signal

\[
(x(t_0), x(t_1), \ldots, x(t_{N-1})).
\]

Note that we don’t sample at \( t_N = \frac{NL}{N} = L \) because, since \( x(t) \) has period \( L \), we would have

\[
x(t_N) = x(t_0).
\]

Complex Sinusoidal Signals

An ordinary sinusoidal function (signal) is a function of the form

\[
x(t) = A\sin(bt) \text{ (or } A\cos(bt))
\]

where \( A \) is called the amplitude and \( b \) is the circular frequency. Recall that the complex exponential is defined by

\[
e^{ibt} = \cos(bt) + i\sin(bt).
\]

We have

\[
\sin(bt) = \frac{e^{ibt} - e^{-ibt}}{2i} \quad \text{and} \quad \cos(bt) = \frac{e^{ibt} + e^{-ibt}}{2},
\]
so we see that anything that can be expressed in terms of sines and cosines can also be expressed in terms of complex exponentials and vice-versa.

If we want to consider signals having period $L$, we can express them in terms of complex exponentials such as

$$e^{2\pi it/L}$$

which has period $L$. In fact, we can define a collection of complex exponential signals having the property that a whole number of complete periods fits into the interval $[0, L]$ as follows:

$$s_k(t) = e^{i\omega_k t}, 0 \leq k \leq N - 1$$

where

$$\omega_k = \frac{2\pi k}{L}$$

Note that $s_k(t)$ has period

$$T_k = \frac{2\pi}{\omega_k} = \frac{L}{k}$$

so that exactly $k$ complete periods of $s_k(t)$ fit into the interval $[0, L]$. We can define a sampled version of the signal $s_k(t)$ as we did for the general signal $x(t)$ above:

$$s_k = (s_k(t_0), s_k(t_1), \ldots, s_k(t_{N-1}))$$

The Complex Vector Space $\mathbb{C}^N$ and the Inner Product

The space $\mathbb{C}^N$ consists of all complex $N$-tuples $(z_1, z_2, \ldots, z_N)$, where each $z_i$ is a complex number. Addition and scalar multiplication are defined as:

$$(z_1, z_2, \ldots, z_N) + (w_1, w_2, \ldots, w_N) = (z_1 + w_1, z_2 + w_2, \ldots, z_N + w_N)$$

$$\alpha(z_1, z_2, \ldots, z_N) = (\alpha z_1, \alpha z_2, \ldots, \alpha z_N)$$

and it is easily checked that, with these operations, $\mathbb{C}^N$ is a (complex) vector space.

We define an inner product on $\mathbb{C}^N$ by

$$<x, y> = \frac{1}{N} \sum_{k=1}^{N} x_k \bar{y}_k$$
We say two vectors $x$ and $y$ are **orthogonal** if

$$<x, y> = 0.$$  

The **length** or **norm** of a vector $x \in \mathbb{C}^N$ is

$$|x| = \sqrt{x^*x}.$$  

A set of vectors in $\mathbb{C}^N$ is said to be **orthogonal** if whenever $x$ and $y$ are two different vectors taken from the set, $<x, y> = 0$. If, in addition, each vector in the set has length 1, we say the set of vectors is **orthonormal**.

**Lemma 1:** The set of sampled signals $s_k$, $0 \leq k \leq N-1$ is an orthonormal basis for $\mathbb{C}^N$.

**Proof:** Since there are $N$ vectors in the set, and since an orthonormal set is automatically linearly independent, it is enough to prove that the set is orthonormal. First, we show that each $s_k$ is a unit vector.

$$<s_k, s_k> = \frac{1}{N} \sum_{j=0}^{N-1} e^{i\omega_j k} e^{-i\omega_j l} = \frac{1}{N} \sum_{j=0}^{N-1} |e^{i\omega_j l}|^2 = \frac{1}{N} \sum_{j=0}^{N-1} 1 = \frac{1}{N} N = 1.$$  

Now we assume $p \neq q$ and both $p$ and $q$ are among $0, 1, \ldots, N-1$ and we show that

$$<s_p, s_q> = 0.$$  

We have

$$<s_p, s_q> = \frac{1}{N} \sum_{k=0}^{N-1} e^{i\omega_k p} e^{-i\omega_k q}.$$  

Now

$$\omega_p k = \frac{2\pi p k L}{N} = \frac{2\pi p k}{N} \quad \text{and} \quad \omega_q k = \frac{2\pi q k}{N}$$

so

$$i\omega_p k - i\omega_q k = \frac{2\pi i(p - q)k}{N}.$$  

Note that

$$-N < p - q < N$$

so that
Thus,
\[
< s_p, s_q > = \frac{1}{N} \sum_{k=0}^{N-1} e^{i \omega k} e^{-i \omega k} = \frac{1}{N} \sum_{k=0}^{N-1} e^{2 \pi i (p-q) k} = \frac{1}{N} \sum_{k=0}^{N-1} \left( e^{\frac{2 \pi i (p-q)}{N}} \right)^k.
\]

This last sum is a geometric sum with common ratio \( e^{\frac{2 \pi i (p-q)}{N}} \neq 1 \) so the sum equals
\[
1 - \left(\frac{e^{\frac{2 \pi i (p-q)}{N}}}{1 - e^{\frac{2 \pi i (p-q)}{N}}} \right)^N.
\]

But the numerator is 0 because
\[
\left( e^{\frac{2 \pi i (p-q)}{N}} \right)^N = e^{2 \pi i (p-q)} = 1.
\]

Therefore, \( < s_p, s_q > = 0 \), as asserted.

**Theorem 1:** Let \( x \in \mathbb{C}^N \). Then \( x \) can be expressed in terms of the basis vectors \( s_0, s_1, \ldots, s_{N-1} \) as
\[
x = \sum_{k=0}^{N-1} \langle x, s_k \rangle s_k.
\]

**Proof:** Since the vectors \( s_0, s_1, \ldots, s_{N-1} \) are a basis for \( \mathbb{C}^N \), it follows that there exist complex numbers \( \alpha_0, \alpha_1, \ldots, \alpha_{N-1} \) such that
\[
x = \alpha_0 s_0 + \alpha_1 s_1 + \cdots + \alpha_{N-1} s_{N-1}.
\]

But then if we take the inner product of both sides with the vector \( s_k \), we get
\[
\langle x, s_k \rangle = \langle \alpha_0 s_0 + \alpha_1 s_1 + \cdots + \alpha_{N-1} s_{N-1}, s_k \rangle = \alpha_0 \langle s_0, s_k \rangle + \alpha_1 \langle s_1, s_k \rangle + \cdots + \alpha_{N-1} \langle s_{N-1}, s_k \rangle = \alpha_k \langle s_k, s_k \rangle = \alpha_k.
\]

This proves the theorem.
The Discrete Fourier Transform (DFT) and Inverse Transform (IDFT)

**Definition:** Given a discrete “signal” \( x \in \mathbb{C}^N \), the Fourier Transform of \( x \) (DFT) is the signal \( \hat{x} \) defined by

\[
\hat{x} = (\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_{N-1}) \text{ where } \hat{x}_k = \left\langle x, s_k \right\rangle.
\]

Thus, to be explicit, the DFT of a signal \( x \in \mathbb{C}^N \) is a vector, denoted \( \hat{x} \), whose k-th component \( \hat{x}_k \) is given by

\[
\hat{x}_k = \frac{1}{N} \sum_{n=0}^{N-1} x_n e^{-i\omega_k t_n}
\]

**Definition:** Given a discrete “signal” \( x \in \mathbb{C}^N \), the Inverse Fourier Transform of \( x \) (IDFT) is the signal \( \tilde{x} \) defined by

\[
\tilde{x} = (\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_{N-1}) \text{ where } \tilde{x}_k = \sum_{n=0}^{N-1} x_n e^{i\omega_k t_n}.
\]

**Theorem 2:** The Inverse Fourier Transform is the inverse of the Fourier Transform. That is, if \( x \in \mathbb{C}^N \), and \( X = \hat{x} \), then \( \tilde{X} = x \).

**Proof:** The k-th component of \( \tilde{X} \) is \( \tilde{X}_k = \sum_{n=0}^{N-1} X_n e^{i\omega_k t_n} \). But the sum on the right side of the equation is just the k-th component of the vector

\[
\sum_{n=0}^{N-1} X_n s_n = \sum_{n=0}^{N-1} \hat{x}_n s_n = \sum_{n=0}^{N-1} \left\langle x, s_n \right\rangle s_n.
\]

Thus, by Theorem 1,

\[
\tilde{X} = \sum_{n=0}^{N-1} \left\langle x, s_n \right\rangle s_n = x,
\]

and the theorem is proved.

**Example 1:** Suppose \( x \) is a sampled signal of length \( N \) given by

\[
x_n = \sin\left(\frac{2\pi p}{L} t_n\right)
\]

where \( p \) is an integer. Thus, the period of the signal is
so \( p \) complete periods fit into the interval \([0, L]\). One can show that:

If \( N = 2r \), then:

\[
\hat{x}_p = \frac{-1}{2i}, \quad \hat{x}_{N-p} = \frac{1}{2i}, \quad \text{all other } \hat{x}_k = 0, \quad \text{except when } p = 0 \text{ or } p = r, \quad \text{in which case } \hat{x} = 0.
\]

If \( N \) is odd, then:

\[
\hat{x}_p = \frac{-1}{2i}, \quad \hat{x}_{N-p} = \frac{1}{2i}, \quad \text{all other } \hat{x}_k = 0, \quad \text{except when } p = 0, \quad \text{in which case } \hat{x} = 0.
\]

For instance, here is the output for various \( p \) values for \( N=10 \).

\[
\begin{align*}
p = 0: & \quad \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0\} \\
p = 1: & \quad \{0, -0.5 I, 0, 0, 0, 0, 0, 0, 0, 0.5 I\} \\
p = 2: & \quad \{0, 0, -0.5 I, 0, 0, 0, 0, 0, 0, 0.5 I, 0\} \\
p = 3: & \quad \{0, 0, 0, -0.5 I, 0, 0, 0, 0, 0.5 I, 0, 0\} \\
p = 4: & \quad \{0, 0, 0, 0, -0.5 I, 0, 0.5 I, 0, 0, 0\} \\
p = 5: & \quad \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\} \\
p = 6: & \quad \{0, 0, 0, 0.5 I, 0, -0.5 I, 0, 0, 0, 0\} \\
p = 7: & \quad \{0, 0, 0, 0, 0.5 I, 0, 0, -0.5 I, 0, 0\} \\
p = 8: & \quad \{0, 0, 0.5 I, 0, -0.5 I, 0, 0, 0, 0, 0\} \\
p = 9: & \quad \{0, 0, 0.5 I, 0, 0, 0, 0, 0, -0.5 I, 0\}
\end{align*}
\]

And here is the output for various \( p \) values for \( N=11 \).

\[
\begin{align*}
p = 0: & \quad \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\} \\
p = 1: & \quad \{0, -0.5 I, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.5 I\} \\
p = 2: & \quad \{0, 0, -0.5 I, 0, 0, 0, 0, 0, 0, 0.5 I, 0, 0\} \\
p = 3: & \quad \{0, 0, 0, -0.5 I, 0, 0, 0, 0.5 I, 0, 0, 0\} \\
p = 4: & \quad \{0, 0, 0, 0, -0.5 I, 0, 0, 0.5 I, 0, 0, 0, 0\} \\
p = 5: & \quad \{0, 0, 0, 0.5 I, 0, 0, 0.5 I, 0, 0, 0, 0\} \\
p = 6: & \quad \{0, 0, 0, 0, 0, 0.5 I, 0, 0, -0.5 I, 0, 0\} \\
p = 7: & \quad \{0, 0, 0, 0.5 I, 0, 0, -0.5 I, 0, 0, 0, 0\} \\
p = 8: & \quad \{0, 0, 0.5 I, 0, 0, 0, 0, -0.5 I, 0, 0, 0\} \\
p = 9: & \quad \{0, 0, 0.5 I, 0, 0, 0, 0, 0, -0.5 I, 0\} \\
p = 10: & \quad \{0, 0.5 I, 0, 0, 0, 0, 0, 0, 0, 0, -0.5 I\}
\end{align*}
\]
Transforming a Finite Data Sequence

Suppose $x_0, x_1, \ldots, x_{N-1}$ are numerical values. We can regard them as a sample of a signal sampled at $0, 1, \ldots, N - 1$. Thus, we take $T = 1$ and hence $L = N$. Then we have

$$t_n = n \text{ and } \omega_n = \frac{2\pi n}{N}.$$ 

The basis of complex exponentials that we constructed above now consists of

$$s_k = \left( e^{\frac{2\pi i k 0}{N}}, e^{\frac{2\pi i k 1}{N}}, e^{\frac{2\pi i k 2}{N}}, \ldots, e^{\frac{2\pi i k (N-1)}{N}} \right) \text{ for } 0 \leq k \leq N - 1.$$ 

The definition of the inner product does not change. As before, we have the representation of the sampled signal $x$ in terms of the $s_k$ as

$$x = \sum_{k=0}^{N-1} \langle x, s_k \rangle s_k$$

so that

$$x_n = \sum_{k=0}^{N-1} \langle x, s_k \rangle e^{\frac{2\pi i k n}{N}}$$

where now

$$\hat{x}_k = \langle x, s_k \rangle = \frac{1}{N} \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i k n}{N}}.$$