Finite Differences, Falling Factorials and Summation

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The Forward Difference Operator

Given a function f, the Newton forward difference operator Δ applied to f defines a new function Δf by the rule

$$\Delta f(x) = f(x+1) - f(x)$$

If $(x_n)_{n \geq n_0}$ is a sequence, we define a new sequence $(\Delta x_n)_{n \geq n_0}$ by

$$\Delta x_n = x_{n+1} - x_n .$$

We can apply the Δ operator repeatedly:

$$\Delta^2 f = \Delta(\Delta f)$$

and in general,

$$\Delta^n f = \Delta(\Delta^{n-1} f)$$

Example: Let $x_n = n^3$ for $n \geq 1$. Then

$$\begin{aligned}
\Delta x_n &= (n+1)^3 - n^3 = 3n^2 + 3n + 1 \\
\Delta^2 x_n &= \Delta(3n^2 + 3n + 1) = 3(n+1)^2 + 3(n+1) + 1 - 3n^2 - 3n - 1 \\
&= 6n + 6 \\
\Delta^3 x_n &= \Delta(6n+6) = 6(n+1) + 6 - 6n - 6 \\
&= 6 \\
\Delta^4 x_n &= \Delta(6) = 0
\end{aligned}$$

Exercise: Find $\Delta(2n^2 - n)$ and $\Delta^2(2n^2 - n)$.

Exercise: Let $x_n = \frac{1}{n}$ for $n \geq 1$. Show that $\Delta x_n = \frac{-1}{n(n+1)}$.

Exercise: Let $x_n = 2^n$ for $n \geq 1$. Find Δx_n

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Exercise:

Show that:

1. $\Delta(x_n + y_n) = \Delta x_n + \Delta y_n$
2. $\Delta^r(x_n + y_n) = \Delta^r x_n + \Delta^r y_n$ for $r \geq 1$ (Use induction on r)
3. For any constant c, $\Delta(cx_n) = c\Delta x_n$
4. $\Delta^r(cx_n) = c\Delta^r x_n$ for $r \geq 1$
5. $\Delta x_n = 0$ for all $n \geq 1$ iff $\exists C \in \mathbb{R}$ such that $x_n = C$ for all $n \geq 1$.
6. $\Delta^2 x_n = 0$ for all $n \geq 1$ iff $\exists a, b \in \mathbb{R}$ such that $x_n = an + b$ for all $n \geq 1$.

Exercise: Show that for any positive constant a, $\Delta(a^x) = (a-1)a^x$.

Definition: For $m \geq 0$ an integer, the **falling factorial** $x^{\underline{m}}$, is defined by

$$x^{\underline{0}} = 1,$$
$$x^{\underline{m}} = x(x-1)(x-2)\cdots(x-m+1) \quad \text{(Note: } m \text{ factors)}$$

Exercise: Write down the polynomials $x^{\underline{2}}$, $x^{\underline{3}}$, and $x^{\underline{4}}$.

If we look back at previous examples, we see that ordinary powers do not work particularly nicely with respect to Δ. Thus we have

$$\Delta x^3 = 3x^2 + 3x + 1$$

On the other hand, the falling factorial works in a manner reminiscent of derivatives.

Lemma: $\Delta(x^{\underline{m}}) = mx^{\underline{m-1}}$

Proof: We have

$$\begin{aligned}
\Delta(x^{\underline{m}}) &= (x+1)^{\underline{m}} - x^{\underline{m}} \\
&= (x+1)x(x-1)\cdots(x-m+2) - x(x-1)\cdots(x-m+2)(x-m+1) \\
&= ((x+1)-(x-m+1))x(x-1)\cdots(x-m+2) \\
&= mx(x-1)\cdots(x-m+2) \\
&= mx^{\underline{m-1}}
\end{aligned}$$

This completes the proof. ▌

Exercise: Use induction to show that for $m \geq 0$, $\Delta^m(x^{\underline{m}}) = m!$.

Application To Summation Problems

Suppose that f is a function and we want to evaluate the sum

$$\sum_{k=a}^{b-1} f(k).$$

This means we want to find a formula for the value of this sum.

Strategy: Suppose we can find a function g such that $\Delta g = f$. Then

$$\begin{aligned}\sum_{k=a}^{b-1} f(k) = \sum_{k=a}^{b-1} \Delta g(k) = \sum_{k=a}^{b-1}(g(k+1)-g(k))\\ &= g(a+1)-g(a)+g(a+2)-g(a+1)+\cdots\\ &\qquad +g(b-1)-g(b-2)+g(b)-g(b-1)\\ &= g(b)-g(a)\end{aligned}$$

Exercise: Use induction to give a rigorous proof of the above formula, that is, show that for any integer a and $n > a$,

$$\sum_{x=a}^{n-1} f(k) = g(n)-g(a).$$

Given the above formula, the problem of finding a formula for a sum $\sum_{k=a}^{b-1} f(k)$ is reduced to the problem of finding a function g such that $\Delta g = f$. This is generally not an easy thing to do, but in certain cases we can do it.

Exercise: Find a formula for the sum $\sum_{k=1}^{n} \frac{1}{k(k+1)}$

Falling Factorials and Powers

Note that the falling factorial $x^{\underline{m}}$ is a polynomial in x of degree m, as is the ordinary power x^m. We can express $x^{\underline{m}}$ in terms of powers of x and conversely, we can express any non-negative integer power x^m in terms of the falling factorials.

Example:

$$x^{\underline{0}} = 1$$
$$x^{\underline{1}} = x$$
$$x^{\underline{2}} = x(x-1) = x^2 - x$$
$$x^{\underline{3}} = x(x-1)(x-2) = x^3 - 3x^2 + 2x$$

Therefore,

$$x^0 = 1 = x^{\underline{0}}$$
$$x = x^{\underline{1}}$$
$$x^2 = x^{\underline{2}} + x = x^{\underline{2}} + x^{\underline{1}}$$
$$\begin{aligned} x^3 &= x^{\underline{3}} + 3x^2 - 2x \\ &= x^{\underline{3}} + 3(x^{\underline{2}} + x^{\underline{1}}) - 2x^{\underline{1}} \\ &= x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}} \end{aligned}$$

But note that we can now write each of these power functions as Δ of some function.

Example: Find g such that $x^2 = \Delta g(x)$ and use this to find a formula for the sum

$$\sum_{k=1}^{n} k^2 .$$

Solution: Since, as shown above, $x^2 = x^{\underline{2}} + x^{\underline{1}}$ we can write $x^2 = x^{\underline{2}} + x^{\underline{1}} = \Delta(\tfrac{1}{3}x^{\underline{3}} + \tfrac{1}{2}x^{\underline{2}})$. Therefore,

$$\begin{aligned} \sum_{k=1}^{n} k^2 = \sum_{k=1}^{n} \Delta(\tfrac{1}{3}k^{\underline{3}} + \tfrac{1}{2}k^{\underline{2}}) &= \tfrac{1}{3}(n+1)^{\underline{3}} + \tfrac{1}{2}(n+1)^{\underline{2}} - \tfrac{1}{3}1^{\underline{3}} - \tfrac{1}{2}1^{\underline{2}} \\ &= \tfrac{1}{3}(n+1)n(n-1) + \tfrac{1}{2}(n+1)n - \tfrac{1}{3}1(1-1)(1-2) - \tfrac{1}{2}1(1-1) \\ &= \tfrac{1}{3}(n+1)n(n-1) + \tfrac{1}{2}(n+1)n \\ &= \frac{2(n+1)n(n-1) + 3(n+1)n}{6} \\ &= \frac{(n+1)n(2(n-1)+3)}{6} \\ &= \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

Thus, we have derived the formula

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Exercise: Use methods analogous to what was done in the preceding example to derive formulas for each of the following sums.

1. $\sum_{k=1}^{n} k$
2. $\sum_{k=1}^{n} k^3$

Note that by proceeding as above we can express any positive integer power x^m, as a linear combination of the falling factorials $x^{\underline{k}}, 0 \le k \le m$. Then we can "integrate" to find a function g such that $x^m = \Delta g(x)$. Therefore, we can find a formula for the sum

$$\sum_{k=1}^{n} k^m .$$

A Property of the Difference Operator

Exercise: Show by direct calculation that $\Delta^3(3n^2 - 4n + 1) = 0$

Exercise: Show by direct calculation that $\Delta^2(n^2 + 2n + 1)$ is a non-zero constant.

Since $\Delta(x^{\underline{m}}) = mx^{\underline{m-1}}$ and $\Delta(x^{\underline{0}}) = 0$, it follows that if $f(x)$ is a linear combination of the terms $x^{\underline{k}}$ for $0 \le k < r$ and $m \ge r$, then

$$\Delta^m f(x) = 0 .$$

But we have also argued that any for integer $k \ge 0$, the power function x^k can be written as a linear combination of the terms $x^{\underline{j}}$ where $0 \le j \le k$. Therefore we may state:

Theorem: Let $p(x)$ be a polynomial of degree less than m and define a sequence $(x_n)_{n \ge 1}$ by $x_n = p(n)$, then $\Delta^m x_n = 0$.

We now prove the converse of this statement.

Theorem: Suppose we are given a sequence $(x_n)_{n\geq 1}$ and suppose m is a positive integer such that $\Delta^m x_n = 0$. Then there is a polynomial $p(x)$ of degree less than m such that $x_n = p(n)$.

Proof: We proceed by induction on m. To prove the case $m=1$, we observe that if $\Delta x_n = 0$, then $x_{n+1} - x_n = 0$ for all $n \geq 1$, so $x_n = x_1$ for all $n \geq 1$, that is, $x_n = p(n)$ where $p(x) = x_1$ is a polynomial of degree <1.
Now suppose $m \geq 1$ and the statement is true for m. We shall prove it for $m+1$. So suppose $(x_n)_{n\geq 1}$ is a sequence and that $\Delta^{m+1} x_n = 0$. We must show there is a polynomial $p(x)$ of degree $\leq m$ such that $x_n = p(n)$ for $n \geq 1$. Now we have

$$0 = \Delta^{m+1} x_n = \Delta(\Delta^m x_n)$$

so by the case $m=1$ proved above, there is a constant C such that $\Delta^m x_n = C$ for $n \geq 1$. Define a sequence $(y_n)_{n\geq 1}$ by $y_n = \frac{C}{m!} n^m$. Then it follows from Exercise 8 that $\Delta^m y_n = C$. Then the sequence $(z_n)_{n\geq 1}$, defined by $z_n = x_n - y_n$ satisfies $\Delta^m z_n = 0$. By our induction hypothesis, there is a polynomial $q(x)$ of degree less than m such that $z_n = q(n)$. Therefore, we have

$$x_n = y_n + q(n).$$

But we know that n^m is a polynomial of degree m in the variable n and hence y_n is a polynomial in n of degree $\leq m$ (we can't say y_n has degree equal to m because of the possibility that $C = 0$). Therefore, there is a polynomial $v(x)$ of degree $\leq m$ such that $x_n = v(n)$ as desired. This completes the inductive step and the proof. ▌