Review of Summation Notation

If \( a_1, a_2, \ldots, a_{10} \) are ten real numbers we can write their sum as

\[ a_1 + a_2 + \cdots + a_{10} \]

or, more compactly, in “summation notation

\[ \sum_{i=1}^{10} a_i \]

\[ \sum \] is called “the summation sign”.
\( i \) is called “the summation index”.
1 and 10 are called “the lower and upper limits of summation”.

Examples:

\[ \sum_{j=1}^{5} (j + 1) = (1 + 1) + (2 + 1) + (3 + 1) + (4 + 1) + (5 + 1) = 20 \]

\[ \sum_{k=3}^{6} k^2 = 3^2 + 4^2 + 5^2 + 6^2 = 86 \]

If \( a_k \) denotes some number for each value of \( k \) between 1 and \( n \), then the sum

\[ a_1 + a_2 + \cdots + a_n \]

can be written as

\[ \sum_{k=1}^{n} a_k \].

Note that sums need not start with \( k = 1 \). When writing examples, we often use \( k = 1 \) just to be definite, but we don’t have to.

Two important identities hold for summations:

- \[ \sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \]

Note that this just says

\[ (a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n) = (a_1 + a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_n) \]

which is certainly true.
• \[ \sum_{k=1}^{n} c a_k = c \sum_{k=1}^{n} a_k \]

This just says
\[ ca_1 + ca_2 + \cdots + ca_n = c(a_1 + a_2 + \cdots + a_n) \]

which is also evident.

**Example:** Consider the matrices
\[
A = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 4 & 1 & 0 & 2 \\ 2 & 2 & 9 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 5 & 3 \\ -2 & 4 & -2 & 1 \\ 0 & 5 & 3 & 1 \\ 5 & 5 & 3 & 1 \end{bmatrix}
\]

Denote the entries of \( A \) by \( a_{ij} \), so that, for instance \( a_{12} = -1 \) and \( a_{33} = 9 \). Also, denote the entries of \( B \) by \( b_{ij} \), so that \( b_{23} = -2 \).

1. Find the numerical value of the (2,3) entry of the product matrix \( AB \).

Answer: \((4)(5) + (1)(-2) + (0)(3) + (2)(3) = 24\)

2. Express the value of that entry in terms of \( a_{ij} \) and \( b_{ij} \), using summation notation.

Answer: \( \sum_{k=1}^{4} a_{2k} b_{k3} \)

**Exercises:**
1. Let \( c_{ij} = 4(i + 2j) \).
   a. Find the value of \( \sum_{j=1}^{3} c_{ij} \) when \( i = 2 \).
   b. Find the value of \( \sum_{i=1}^{4} c_{ii} \)

2. Write the sum of the first 25 integers in summation notation.

3. Suppose \( \mathbf{u} = \begin{bmatrix} b_1, b_2, \ldots, b_p \end{bmatrix} \) is a \( 1 \times p \) matrix. Express the sum of the entries of \( \mathbf{u} \) using summation notation.
4. Let \( A = \begin{bmatrix} a_{ij} \end{bmatrix} \) be an \( m \times n \) matrix.

   a. Express the sum of the elements in the first row of the matrix using summation notation.

   b. Express the sum of the elements in the first column of the matrix using summation notation.

   c. Suppose \( 1 \leq i \leq m \). Express the sum of the elements in row \( i \) of the matrix using summation notation.

   d. Suppose \( 1 \leq j \leq n \). Express the sum of the elements in column \( j \) of the matrix using summation notation.

5. Suppose \( A = \begin{bmatrix} a_{ij} \end{bmatrix} \) is a \( 3 \times 4 \) matrix and \( B = \begin{bmatrix} b_{ij} \end{bmatrix} \) is a \( 4 \times 5 \) matrix.

   a. What are the dimensions of the matrix product \( AB \)?

   b. Express the \((2,4)\)-element of \( AB \) using summation notation in terms of the entries of \( A \) and the entries of \( B \).

6. Let \( A = (a_{ij}) \) be an \( m \times n \) matrix and let \( B = (b_{ij}) \) be an \( n \times p \). If \( 1 \leq i \leq m \) and \( 1 \leq j \leq p \) which of the following expressions represents the \((i, j)\) entry of the matrix product \( AB \)?

   a. \( \sum_{k=1}^{n} a_{ik} b_{jk} \)

   b. \( \sum_{k=1}^{n} a_{ik} b_{kj} \)

   c. \( \sum_{k=1}^{n} a_{ij} b_{jk} \)
Answers:

1.
   a. 72
   b. 120

2. $\sum_{k=1}^{25} k$

3. $\sum_{k=1}^{p} b_k$

4.
   a. $\sum_{j=1}^{n} a_{ij}$
   b. $\sum_{i=1}^{m} a_{ii}$
   c. $\sum_{j=1}^{n} a_{ij}$
   d. $\sum_{i=1}^{m} a_{ij}$

5.
   a. $3 \times 5$
   b. $\sum_{k=1}^{4} a_{2k} b_{k4}$

6. b