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Rough or/and Fuzzy Handling of Uncertainty ?

Rough sets and fuzzy sets (Dubois & Prade, 1991)

„Rough sets have often been compared to fuzzy sets, sometimes with a view to introduce them as competing models of imperfect knowledge.

Such a comparison is misfounded. Indiscernibility and vagueness are distinct facets of imprefect knowledge.

So, one theory is quite distinct from the other, and they display a natural complementarity.

And it would be nice to use both conjointly, since vagueness and coarseness are sometimes interacting."

Fuzzy set extensions of rough sets

- Nakamura & Gao 1991; Dubois & Prade 1992; Lin 1992; Słowiński 1995; Pal 1996; Słowiński & Stefanowski 1996; Yao 1997; Cattaneo 1998; Morsi & Yakout 1998; Greco, Matarazzo & Słowiński 1999, 2000; Thiele 2000; Inuiguchi & Tanino 2002; Polkowski 2002, Greco, Inuiguchi & Słowiński 2002, Radzikowska & Kerre 2003; Wu, Mi & Zhang 2003; ...

- The fuzzy extensions of Pawlak's definition

  of lower and upper approximations use fuzzy connectives

  (t-norm, t-conorm, fuzzy implication)

- In general, however, fuzzy connectives depend on cardinal properties

  of membership degrees,

  i.e. the result is sensitive to order preserving transformation of

  membership degrees

Fuzzy set extensions of rough sets

- **A natural question arises**: is it reasonable to expect from membership degree a cardinal meaning instead of ordinal only?
- In other words, which one of the two statements a human expert is able to express in a meaningful way:

  „object *x* belongs to fuzzy set *X* more likely than object *y*"

  or

  „object *x* belongs to fuzzy set *X* two times more likely than object *y*"?

Dominance-based (monotonic) Rough Approximation of a Fuzzy Set

- The dominance-based rough approximation of a fuzzy set avoids arbitrary choice of fuzzy connectives and not meaningful operations on membership degrees
- Approximation of knowledge about *Y* using knowledge about *X* is based on positive or negative relationships between premises and conclusions, called *gradual rules*, i.e.:

  i) „the more *x* is *X*, the more it is *Y*" (positive relationship)

  ii) „the more *x* is *X*, the less it is *Y*"   (negative relationship)

- Example:

  „the larger the market share of a company, the larger its profit"

  „the larger the debt of a company, the smaller its profit"

S.Greco, M.Inuiguchi, R.Słowiński: Fuzzy rough sets and multiple-premise gradual decision rules. *Int. Journal of Approximate Reasoning*, 41 (2005) 179-211

Distinguishing uncertainty (fuzziness) & granularity (indiscernibility)

- Given universe U and concept $X \subseteq U$, we will represent uncertainty (fuzziness) by means of disjoint sets (I,E), where
  - I represents the „surely yes" part
  - E represents the „surely no" part
- We will also take into account granularity (indiscernibility) by means of a new operator, Pawlak operator, that assigns to any pair (I,E) a new pair ($\underline{R}(I)$, $\underline{R}(E)$), where
  - $\underline{R}(I)$ represents the lower approximation of the „surely yes"
  - $\underline{R}(E)$ the lower approximation of the „surely no"

S. Greco, B. Matarazzo, R. Słowiński: Distinguishing vagueness from ambiguity by means of Pawlak-Brouwer-Zadeh lattices. [In]: *IPMU 2012*

De Morgan Brouwer-Zadeh lattice
(Cattaneo & Nisticò 1989)

Quasi Brouwer-Zadeh Distributive Lattices (1)

- A system $<\Sigma,\wedge,\vee,',{}^{\sim},0,1>$ is quasi Brouwer-Zadeh distributive lattice if the following properties hold
- $<\Sigma,\wedge,\vee,0,1>$ is a distributive lattice
- $' : \Sigma \to \Sigma$ is a Kleene complementation, that is for all $a,b \in \Sigma$
  - (K1) $a'' = a$
  - (K2) $(a \vee b)' = a' \wedge b'$
  - (K3) $a \wedge a' \leq b \vee b'$

Quasi Brouwer-Zadeh Distributive Lattices (2)

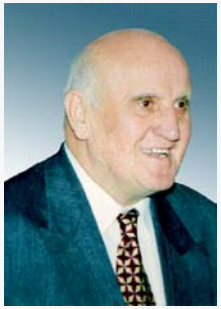
- $\sim : \Sigma \to \Sigma$ is a Brouwer complementation, that is for all $a,b \in \Sigma$
  - (B1) $a \wedge a^{\sim\sim} = a$
  - (B2) $(a \vee b)^{\sim} = a^{\sim} \wedge b^{\sim}$
  - (B3) $a \wedge a^{\sim} = 0$
- (weak interconnection) for all $a \in \Sigma$, $a^{\sim} \leq a'$
- A quasi Brouwer-Zadeh lattice is a Brouwer-Zadeh lattice

  if stronger interconnection rule is satisfied:

  (strong interconnection) for all $a \in \Sigma$, $a^{\sim\sim} = a^{\sim\prime}$
- De Morgan property of B-Z lattice: $(a \wedge b)^{\sim} = a^{\sim} \vee b^{\sim}$

New proposal

Pawlak-Brouwer-Zadeh lattices



Pawlak operator: intuition

- Intuitively, unary operator $^A: \Sigma \to \Sigma$ should be represented as

$$(I,E)^A = (\underline{R}(I), \underline{R}(E))$$

in a de Morgan Brouwer-Zadeh distributive lattice

Pawlak operator: formal definition

- In a de Morgan Brouwer-Zadeh distributive lattice, the unary operator ^{A}: $\Sigma \rightarrow \Sigma$ satisfies the following properties: for all $a,b \in \Sigma$
  - A1) $a^{A\prime} = a^{\prime A}$
  - A2) $a \leq b$ implies $b^{A\sim} \leq a^{A\sim}$
  - A3) $a^{A\sim} \leq a^{\sim}$
  - A4) $0^A = 0$
  - A5) $a^{\sim} = b^{\sim}$ implies $a^A \wedge b^A = (a \wedge b)^A$
  - A6) $a^A \vee b^A \leq (a \vee b)^A$

Pawlak operator: formal definition

- A7) $a^{AA}=a^{A}$
- A8) $a^{A\sim A}=a^{A\sim}$
- A9) $(a^{A}\wedge b^{A})^{A}=a^{A}\wedge b^{A}$

Dominance-based Rough Set Approach to Decision under Risk and Uncertainty

DRSA to decision under risk and uncertainty

- $\boldsymbol{ST}=\{st_1, st_2, st_3, \ldots\}$ – set of elementary states of the world
- Pr – a priori probability distribution over $\boldsymbol{ST}$
  e.g.: $pr_1=0.25$, $pr_2=0.30$, $pr_3=0.35$, ...
- $\boldsymbol{A}=\{A_1, A_2, A_3, A_4, A_5, A_6, \ldots\}$ – set of acts
- $\boldsymbol{X}=\{0, 10, 15, 20, 30, \ldots\}$ – set of possible outcomes (gains)
- $\boldsymbol{Cl}=\{Cl_1, Cl_2, Cl_3, \ldots\}$ – set of quality classes of the acts,
  e.g.: Cl_1=bad acts, Cl_2=medium acts, Cl_3=good acts
- $\rho(A_i,\pi)=x$ means that by act A_i one can gain *at least x* with at least probability $\pi=Pr(\boldsymbol{W})$, where $\boldsymbol{W}\subseteq\boldsymbol{ST}$ is an event
- There is a partial preorder on probabilities π of events
- Act A_i ***stochastically dominates*** A_j iff $\rho(A_i,\pi) \geq \rho(A_j,\pi)$ for each probability π

DRSA to decision under risk and uncertainty

- Preference information given by a Decision Maker:

  assignment of some acts to quality classes

- Example:

π/*Act*	A_1	A_2	A_3	A_4	A_5	A_6
.25	30	20	20	20	20	20
.35	10	20	20	20	20	20
.40	10	20	20	20	20	20
.60	10	20	15	15	20	20
.65	10	20	15	15	20	20
.75	10	20	0	15	10	20
1	10	0	0	0	10	10
Class	good	medium	medium	bad	medium	good

DRSA to decision under risk and uncertainty

- Decision rules induced from rough approximations of quality classes

$$\text{if } \rho(A_i, 0.75)\geq 20 \text{ and } \rho(A_i, 1)\geq 10, \text{ then } A_i \in Cl_3^{\geq} \qquad (A_6)$$

"if the probability of gaining at least 20 is ≥0.75, and the probability of gaining at least 10 is 1, then act A_i is at least good"

$$\text{if } \rho'(A_i, 0.25)\leq 20 \text{ and } \rho'(A_i, 0.75)\leq 15, \text{ then } A_i \in Cl_2^{\leq} \qquad (A_3, A_4, A_5)$$

"if the probability of gaining at most 20 is ≥0.25, and the probability of gaining at most 15 is ≥0.75, then act A_i is at most medium"

- Generalization:
  DRSA for decision under risk with outcomes distributed over time
  (decision under uncertainty and time preference)

Greco S., Matarazzo B., Slowinski R., Dominance-based rough set approach to decision under uncertainty and time preference. *Annals of Operations Research*, 176 (2010) 41-75

DRSA-PCT to decision under risk and uncertainty

- Decision rules induced from rough approximations of binary preference relations on pairs of acts A_i, A_j :

„if the probability of gaining at least 20$ more is ≥0.75,
and the probability of gaining at least 10$ more is 1,
then act A_i is better than act A_j"

„if the probability of gaining at least 10$ less is ≥0.5,
and the probability of gaining at least 5$ less is ≥0.8,
then act A_i is worse than act A_j"