de Finetti's heritage: a unifying tool for handling uncertainty due to different causes

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Uncertainty

There is uncertainty every time there is a lack of information about an event

what causes this lack of information?

- the event is the possible result of a random experiment
- the event will occur (or not) in the future
- the event or its contrary is occurred but one does not know which of them
- the (complete) information is about a set of events different from that of interest
- the information about the event is imprecise
-the information about the event is vague or expressed in natural language
-……

The model

Is there a mathematical model for handling the uncertainty due to different causes and the information coming from different sources?

Following de Finetti’s heritage it is possible to develop a general and robust model which acts as a frame able to assure consistency of different approaches.

Miles stones

EVENTS: an *event* can be singled-out by a (nonambiguous) *proposition* *E*, that is a statement that can be either *true* or *false*.

Directing attention to events as subsets of the sample space may be unsuitable for many real situations, which make instead very significant both to give to events a more general meaning and to assume that there is **no given specific structure** for the family $\mathcal{E}$ where an uncertainty measure is assessed

Miles stones

This (general) framework does not require to distinguish *available* information (for example, evidence coming from statistical data) from any other *potential* or assumed (i.e., even if not yet observed) information.

In this way, by suitably exploiting the aforementioned "status" of all available information as being of the same quality and nature, we can put on the same frame all the relevant events, looked on as *propositions*.

Conditional events

For a conditional event E|H it is essential to regard the conditioning event H not just as a given fact, but as an (uncertain) event, for which the knowledge of its truth value is not required

(as a “variable”, with the same state of E)

Direct definition

- Define conditional probability P(.|.) as a primitive concept, by means of some properties (axioms).

This removes any problems related to zero-measure for conditioning events and permits to express conditional probability evaluations without know the unconditional ones.

Conditional probability

$\mathcal{G} = \mathcal{E} \times \mathcal{H}^0$, $\mathcal{H}^0 = \mathcal{H} / \{\Phi\}$

$\mathcal{E}$ Boolean algebra, $\mathcal{H}$ additive set , $\mathcal{H} \subseteq \mathcal{E}$

A function P: $\mathcal{G} \rightarrow [0,1]$ is a **conditional probability** if satisfies the following axioms

- P(E|H) = P (E $\wedge$ H|H), for every E$\in \mathcal{E}$ and H$\in \mathcal{H}^0$
  for any given H $\in \mathcal{H}^0$
- P (•|H) is a (finitely additive) probability on $\mathcal{E}$
- for every E$\in \mathcal{E}$ and K, H $\in \mathcal{H}^0$, K$\wedge$H $\neq \Phi$

  P(E$\wedge$K|H)=P(K|H) P(E|K$\wedge$H)

coherent conditional probability

A function P(·|·) defined on an arbitrary set of *conditional* events $\mathcal{C}$ is a coherent conditional probability if admits an extension on a set $\mathcal{E}\times\mathcal{H}^0 \supseteq \mathcal{C}$ which is a conditional probability

$\mathcal{E}$ Boolean algebra,

$\mathcal{H}$ additive set , $\mathcal{H}\subseteq\mathcal{E}$ and

$\mathcal{H}^0 = \mathcal{H}/\{\Phi\}$,

Fundamental result

Theorem - Let $\mathcal{C}$ be a family of conditional events and P a corresponding assessment; then there exists a (possibly not unique) coherent extension of P to an arbitrary family $\mathcal{K} \supseteq \mathcal{C}$, if and only if P is coherent on $\mathcal{C}$.

Characterization

There are in the literature many characterizations of coherent conditional probability assessments:

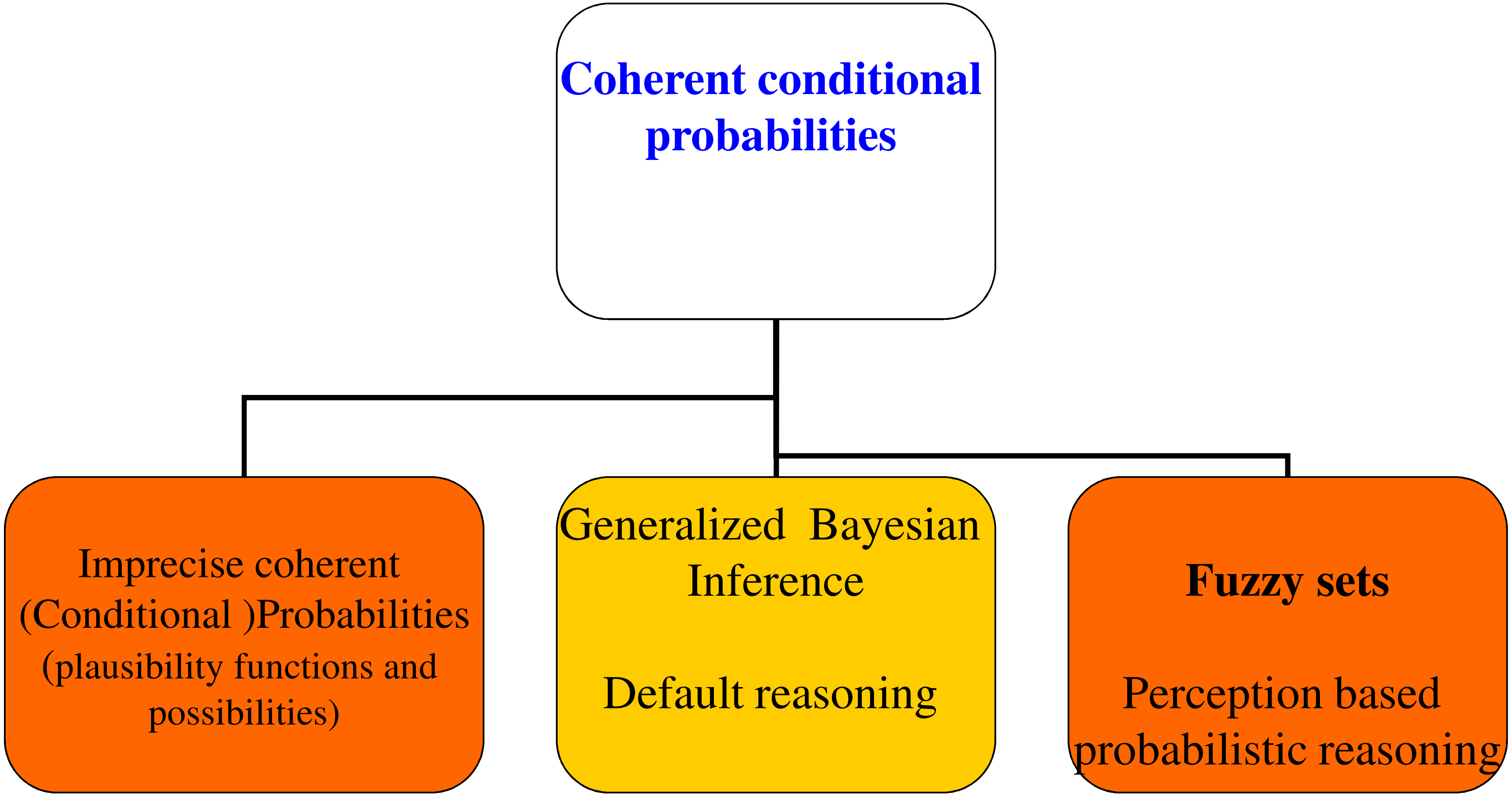
-in terms of (coherent) betting scheme (Williams, Regazzini,...)

- in terms of penalty (Bruno-Gilio)

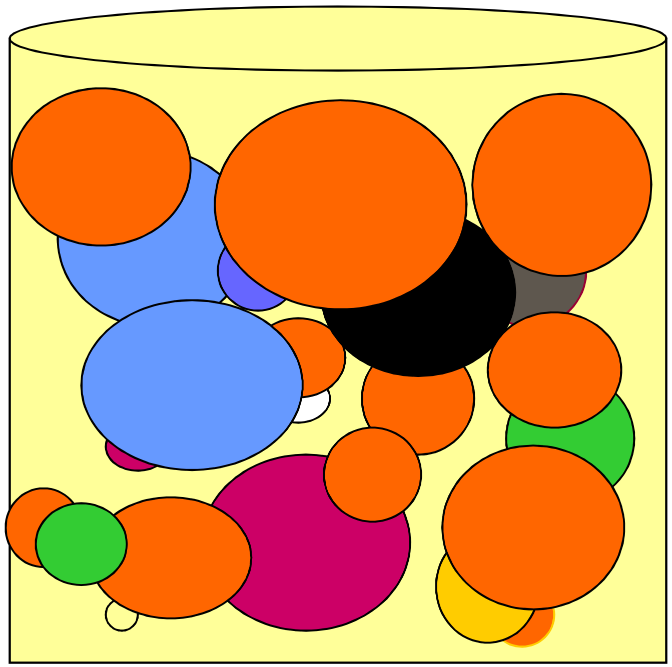
- in terms of the existence (in any finite subset) of a class of coherent (unconditional) probabilities agreeing with the conditional assessment, which are the solutions of a sequence of linear systems (C.-S.)

Characterizations provide effective tools for checking coherence and for determining and ruling the coherent extensions

The model of reference



Unknown composition

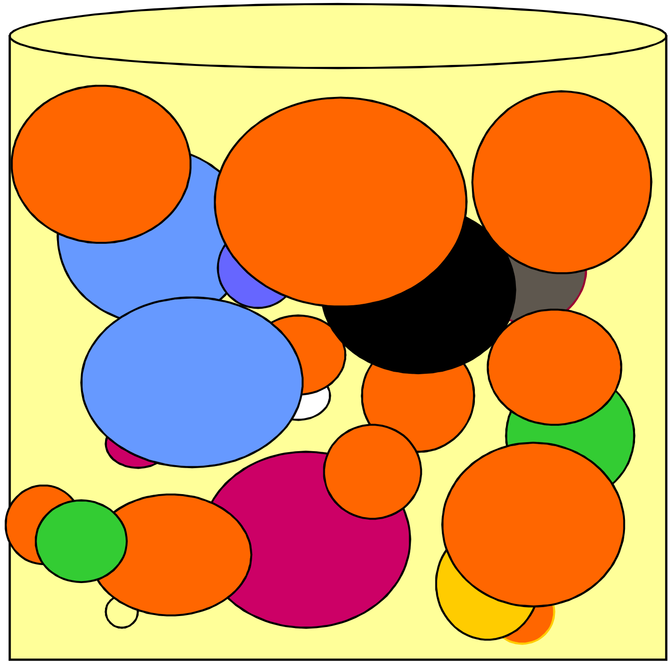


An urn whose composition is not completely known

Consider the experiment consisting on drawing a ball by the urn

and the problem: which is the most probable size of the ball under the hypothesis that the size is ***larger than k***?

Partial information on composition



We do not know the percentage of $s_1, \ldots, s_k$ but, for instance, that of

$B_1 = \{s_1, s_2, s_4\}$, $B_2 = \{s_2, s_5\}$,
$B_1 = \{s_3, s_6\}$, $B_1 = \{s_3, s_7, s_8\}$,....

By coherence we can give an answer to the problem.

Coherent extensions

The properties satisfied by the lower and upper envelope of all the coherent extensions strictly depend on the logical constraints among the events where we have information and those of interest (in which the information is inferred)

-In general are only lower and upper probabilities

-If both the families are partitions the lower envelope is a belief function and the upper a plausibility

-If in particular the partitions are weakly logically independent we obtain a necessity and a possibility

-……….

Coherent likelihood

Coherent conditional probability regarded as function of the conditional event P(E|.).

- If the domain is a partition then any assessment is coherent [P(E|H)=1 if $H \subseteq E$ and P(E|H)=0 if $H \wedge E = \Phi$]
- Coherence forces to have as composition rule for disjunction a weighed mean with weights possibly equal to zero except one (=1)
- Is monotone wit respect to $\subseteq$ if and only if is a possibility

A model for fuzzy sets

We refer to the interpretation (C.- S. 2002) of the membership of a fuzzy subset as a coherent conditional probability, regarded as a

function of the conditional events, which coincides with a (coherent) ***likelihood***, from a syntactic point of view, in a coherent probabilistic setting.

The bridge

X is a variable with range C_X

A_x the event $\{X=x\}$

φ a related property

E_φ the event "You claim (that X is) φ"

EXAMPLE:

X = "size"   φ = "large"

The bridge

A fuzzy subset E_φ^* of C_X is the pair

$$E_\varphi^* = \{E_\varphi , P(E_\varphi | A_x)\}$$

Where $P(E_\varphi|A_x)$ is a coherent conditional probability

$$P(E_\varphi|A_x) = \mu_\varphi(x)$$

is the membership function

The bridge....(first step)

So in this model a **membership function**

of the fuzzy subset E_{φ}^*

$$\mu_{\varphi}(x) = P(E_{\varphi}|A_x)$$

is a **measure** of how much is probable that You, are willing to claim that the variable X has the property φ, when X assumes the different values of its range.

The freedomness in assessing a coherent likelihood assures that there is no limitation for the values of the membership.

Operations and t-norms

Given two fuzzy subsets $E_\varphi{}^*$ and $E_\psi{}^*$, with E_φ and E_ψ logically independent, the definitions of the binary operations of union and intersection related to (archimedean t-norms and t-conorms) and

that of complementation are obtained directly by using the rules of coherent conditional probability

Fuzzy logic

All the t-norms and t-conorms satisfying Frank equation

$$S(x,y) = x + y - T(x,y)$$

are coherent extensions of P to the new events.

In particular:

max **and** ***min***

probabilistic sum **and** ***product***

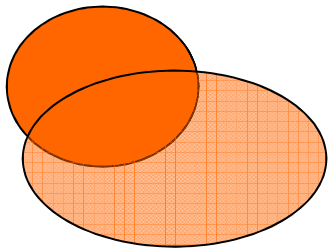
L t-norm **and** ***L t-conorm***

Fuzzy logic

Negation (complementary)

$(E_\varphi{}^*)^c = E_{\neg\varphi}{}^* \neq (\neg E_\varphi)^*$

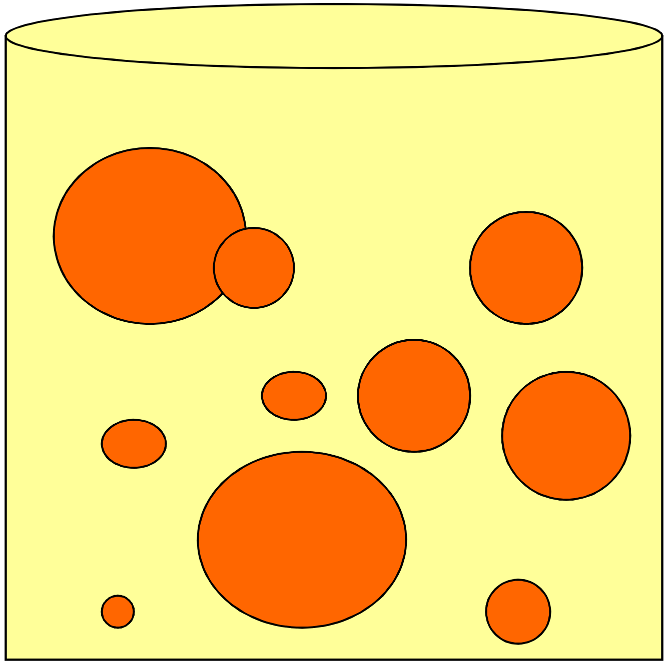
In fact they are logically independent



So

$$E_\varphi{}^* \cup E_{\neg\varphi}{}^* \neq E_\varphi{}^* \cup (\neg E_\varphi)^* = X$$

First problem



Consider an experiment consisting on drawing a ball by an urn whose composition is known

<u>PROBLEM:</u>

which is the probability to obtain a ***large ball***?

In this case

X is the "size of the ball" $C_X = \{s_1, \ldots, s_k\}$

A_i the event $\{X = s_i\}$

L (φ) "large"

E_L the event "You claim that the size is large"

Trivial computation

we have $P(E_L|A_i)$ and $P(A_i)$ for every s_i

Since a likelihood and any probability distribution on the same partition are globally coherent

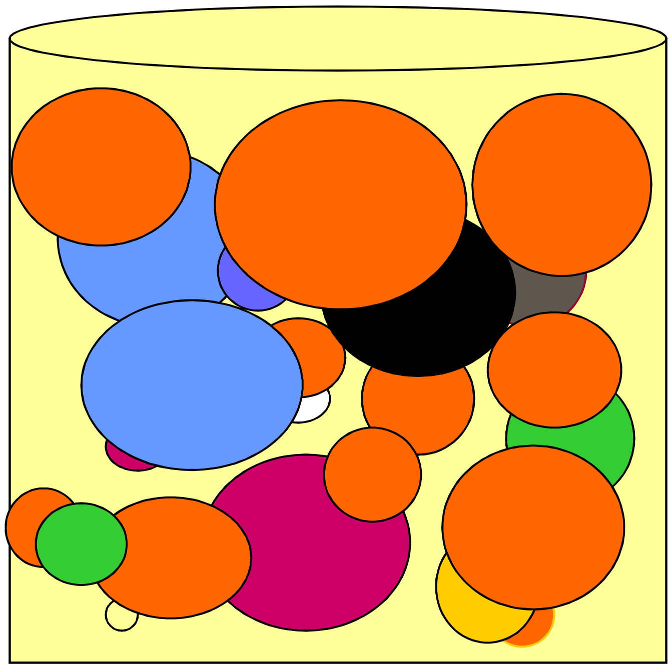
We can compute

$$P(E_L) = \Sigma_i P(E_L|A_i) P(A_i)$$

For us : PROBABILITY OF AN EVENT

For Zadeh: PROBABILITY OF A FUZZY EVENT

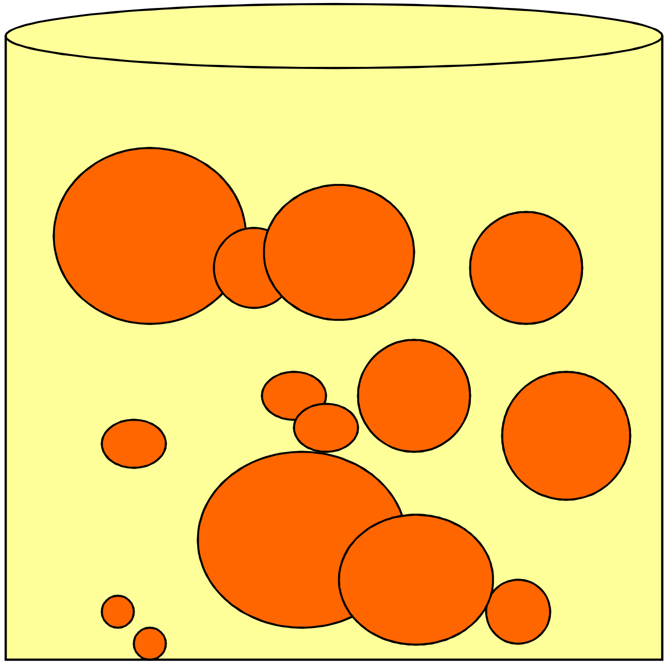
Fuzzy hypothesis



Consider the experiment consisting on drawing a ball by an urn without showing the result.

and consider the problem:
which is the most probable size of the ball under the hypothesis that the size is ***large***?

Uncertainty and vagueness



Consider an experiment consisting on drawing a ball by the urn, whose composition is known, without showing the result.

Suppose to consider the following information (hypothesis) on the result: the size of the ball is large

<u>PROBLEM:</u>

Which is the most probable (size of the) ball?

In this case

We have a coherent conditional probability (likelihood)

$\mu_L(s_i) = P(E_L|A_i)$ for every s_i

We have also, for every s_i, the (prior) probability $P(A_i)$ to draw a bal of the size s_i

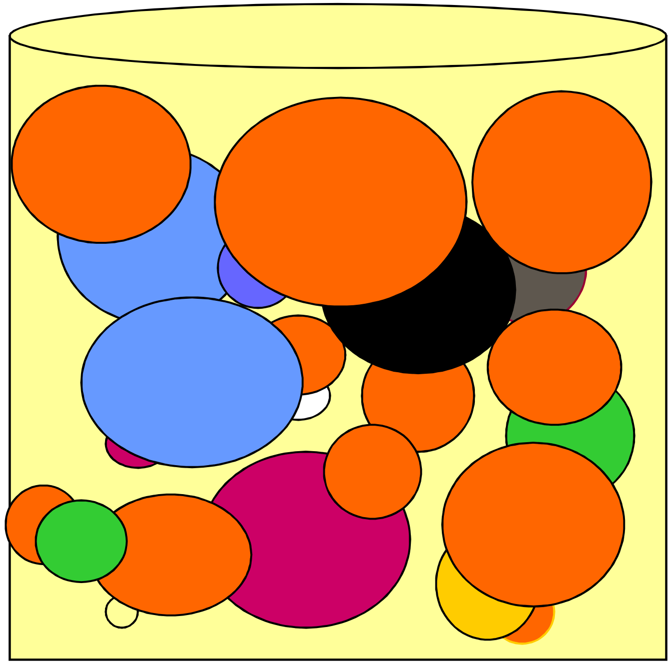
We can compute, by a trivial computation (Bayes rule)

$$P(A_i|E_L) = P(E_L|A_i)\, P(A_i) \,/ \Sigma_i\, P(E_L|A_i)\, P(A_i)$$

A remark

The only remark is semantic: in this Bayesian procedure the prior probability comes from data (is in general is a frequency) and the likelihood is a subjective evaluation (the probability that you claim large the size of the bal in the hypothesis that the size is s_i)

Unknown composition



"Probability to draw a big and dark ball"

In the hypothesis that most balls are big and few are dark