Time-domain modeling of atmospheric turbulence effects on sonic boom propagation

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A 2-D nonlinear time domain computational model of sonic boom propagation has been modified to incorporate the effects of atmospheric turbulence. This model, based on the Nonlinear Progressive wave Equation (NPE) of McDonald and Kuperman, is used to propagate N-waves from the upper turbulent boundary layer (TBL) to the ground through different turbulence realizations. The output of the model provides detailed images of the full wave field at arbitrary heights above the ground, as well as shock profiles at specified locations along the wavefront. Preliminary results show multiple scales of spiking and rounding of shock profiles that are clearly correlated with wavefront focusing and defocusing, respectively. These waveform distortions, as well as their spatial variation along the wavefront, qualitatively match those seen in sonic booms recorded at the ground. These results are also consistent with earlier studies that applied the NPE to weak shocks with rippled wavefronts propagating through a homogeneous medium; these studies demonstrated that the combination of nonlinear and geometric effects arising from focusing wavefronts can produce the variety of distorted sonic boom waveforms observed in test flights. This paper describes the details of the sonic boom propagation code, including the implementation of turbulence effects, and discusses its performance on benchmark problems.

I. Introduction

The future of commercial supersonic transport depends in large part on solving the problem of environmental noise, especially sonic booms. Although recent work has demonstrated the potential of shaped sonic booms, an aspect of the sonic boom problem that continues to receive attention is the variation of shock rise time and peak amplitude that arises from propagation through atmospheric turbulence. Different approaches to understanding the mechanism by which sonic booms are distorted include scattering theory and wave front folding. The former approach yields a statistical description of sonic boom peak pressures and rise times, but does not predict actual wave forms. The mechanism of wavefront focusing and folding has been confirmed by numerical simulations of N-waves with rippled wave fronts in a 2-D homogeneous medium, which produced peaked, multiply-peaked, and rounded shock profiles similar to the variety of observed sonic booms. More recently, it has become feasible with readily available computers to perform time-domain simulations of a sonic boom propagating through a turbulent atmosphere. This paper presents the details of such an effort.

It is well known, from both weak shock theory and measurements, that sonic booms far from the source aircraft, in the absence of atmospheric distortion, have the classic N-wave profile. One advantage of directly simulating a single event sonic boom propagating through turbulence is the ability to follow the evolution of the waveform distortion over time, providing insight regarding the time and length scales over which...
which turbulence contributes to specific deviations from the N-wave profile. Although a particular simulated waveform cannot be expected to precisely match any recorded waveform at the ground, due to the random nature of turbulence, it can be argued that if the simulation incorporates the appropriate physical processes involved in weak shock propagation through atmospheric turbulence, then particular modeled outcomes are representative of actual outcomes. Moreover, if the computational run time of individual realizations is sufficiently small, and the code is run simultaneously on multiple processors, then outcomes can be generated for many realizations of turbulence, providing a data set for computing ensemble averages of quantities of interest, such as shock rise time and peak overpressure, and their correlation to parameters of atmospheric turbulence. Loecey and Sparrow have demonstrated that such a data set could be used to generate filter functions that operate on arbitrarily shaped sonic booms to add the random effects due to turbulent distortion seen on the ground.\textsuperscript{15,16}

The present work makes use of the Nonlinear Progressive-wave Equation, or NPE, which incorporates dissipation, narrow angle refraction or scattering, and nonlinearity up to second order in the acoustic mach number. Developed by McDonald and Kuperman for modeling far-field shock propagation in the ocean,\textsuperscript{17} the NPE is formulated in the time domain, although the acronym is meant to suggest kinship with the widely used frequency domain Parabolic Equation (PE). Both methods model waves that propagate within a narrow angle of a primary axis. Unlike the PE, the NPE readily accommodates nonlinear terms and is useful for modeling pulse propagation; it was designed specifically to study nonlinear wave behavior at caustics.\textsuperscript{17}

The NPE is mathematically equivalent\textsuperscript{18} to the KZK equation, which is also used for narrow angle pulse propagation,\textsuperscript{19} but which marches forward in space, rather than time. The NPE also allows for small local perturbations in the sound speed, as well as large-scale sound speed gradients, making it well suited to modeling pulse propagation through a turbulent medium.

It is important to note that the NPE (and KZK) approach is inherently multidimensional, whereby narrow-angle refraction is modeled naturally. This should be distinguished from ray-type or Burgers equation approaches, which are inherently one-dimensional, and in which refractive focusing and defocusing can only be applied in an ad-hoc manner at best. The NPE does not have this limitation. However, the cost of modeling refraction directly is that the NPE is substantially slower than either ray-type or Burgers-type approaches.

Section II of this paper describes how the original NPE has been adapted to model sonic boom propagation through turbulence and provides details about the numerical implementation. Special attention is given to the computational challenges associated with including turbulence. The code was tested on simple benchmark problems to demonstrate the correct modeling of nonlinear and refractive behavior; some results are shown and discussed in section III.

II. Numerical implementation of NPE

The original NPE (equation 11 of McDonald and Kuperman\textsuperscript{17}) was written essentially as

\[
\frac{\partial p}{\partial t} + c_0 \frac{\partial p}{\partial x} = - \frac{\partial}{\partial x} \left( c' p + \beta \frac{p^2}{2 \rho c_0} \right) - c_0 \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \int_{\infty}^{x} p(x') dx'
\]  
(1)

where \( p \) is the acoustic pressure, \( x \) is the primary direction of propagation, \( c' \) is the local sound speed perturbation, \( c_0 \) and \( \rho \) are the respective ambient values of sound speed and density, and \( \beta \) is the coefficient of acoustic nonlinearity. The left side of the equation represents one-dimensional linear wave propagation, while each right hand term is a representation of a distinct physical process: refraction, nonlinearity, and diffraction (from left to right). The parabolic approximation, valid for narrow angles from the \( x \) axis, is indicated by the presence of a first order derivative in \( x \) with second order derivates in the transverse directions (\( y \) and \( z \)).

One of the computational advantages of the NPE equation is that the computational domain can be chosen to be a reference frame moving in the \( x \) direction with speed \( c_0 \), so that the waveform evolves over the much longer time scales of nonlinearity and diffraction, rather than propagation. Doing so is equivalent to expressing Eq. (1) in terms of the retarded time \( t \rightarrow \tau = t - x/c_0 \), effectively reducing the left side to a simple time derivative. Such a moving reference frame enables the use of relatively large time step values, especially when nonlinear effects dominate, significantly reducing the overall computation time.

As mentioned earlier, the NPE was originally intended for underwater applications. Adapting this equation for sonic boom propagation requires the addition of terms that account for viscous dissipation and molecular relaxation. In order to make use of certain numerical techniques (explained below), the equation must be
expressed in conservative form
\[ \frac{\partial p}{\partial t} + \frac{\partial f(p)}{\partial x} + \frac{\partial g(p)}{\partial y} = 0 \]

in which \( f(p) \) and \( g(p) \) represent the flux in the \( x \) and \( y \) directions, respectively. A simple substitution of mass for the acoustic pressure \( p \) in the above equation yields the familiar conservation of mass equation for a two dimensional control volume.

For the NPE to be written in this form, an approximation is made that molecular relaxation processes are in a steady state and can be represented by an increase in the classical dissipation coefficient.\(^{14} \) This condition is strictly true for a planar N wave. Although focusing wave fronts, such as are generated by turbulence, are clearly not in a steady state, the time scale of wave front evolution is sufficiently slower than molecular relaxation processes that the effect of this approximation on shock rise times is expected to be negligible to first order.

Another simplification is that diffraction effects and off-axis propagation are assumed to occur only in the \( y \) direction; the wave field is uniform in the \( z \) direction. This restricts the problem to two dimensions, allowing for much faster computation of the solution. Efforts are ongoing to extend the computational solution to three dimensions.

Thus modified, the NPE used in modeling sonic boom propagation can be put into the following conservative form (in a reference frame moving with speed \( c_0 \))\(^{12} \)
\[ \frac{\partial p}{\partial t} + \frac{\partial}{\partial x} \left( c' + \frac{\beta p^2}{2 \rho c_0} - \delta_{\text{eff}} \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{c_0 \partial p}{2 \partial y \int_{-\infty}^{x} p dx'} \right) = 0 \]

where \( \delta_{\text{eff}} \) is the effective dissipation coefficient that incorporates both classical (thermoviscous) dissipation and steady-state molecular relaxation appropriate for sonic booms.

The conservative form of Eq. (3) enables the use of a numerical scheme that employs a conservation law within each cell of the computational grid, which ensures correct physical behavior in the vicinity of a shock. This scheme is a flux corrected transport (FCT) algorithm that selectivity applies first order or higher order upwind differencing, depending on the behavior of the fluxes surrounding each grid point. The net result is that high order accuracy is obtained where the solution is smooth, while the dissipative first order differencing is applied only at steep gradients. In addition to eliminating the need for an artificial viscosity term, the FCT method guarantees monotonicity in the solution. This means that no new extrema, such as Gibbs oscillations, are allowed to form solely from the transport equation in either direction alone. This was the strategy employed by McDonald and Kuperman to solve the original NPE, but only to advance the solution in the \( x \) direction; they used the Crank-Nicolson method to integrate the diffraction part of the equation, splitting the time step between the forward and transverse directions.\(^{20} \)

The version of FCT applied here is that of McDonald and Ambrosiano,\(^{21} \) who adapted the original algorithm of Boris and Book\(^{22} \) to hyperbolic equations and improved it by employing only one-sided differences and adding a flux-checking procedure to prevent expansion shocks from forming. Instead of employing FCT only in the \( x \) direction and splitting the time step, the algorithm of McDonald and Ambrosiano is extended here to two dimensions, as developed in reference [12] following the approach of Zalesak.\(^{23} \) The two dimensional FCT algorithm limits the fluxes in the \( x \) and \( y \) directions separately, consistent with the original one dimensional methodology. This allows the solution to develop new extrema due to two dimensional effects, such as focusing and scattering, while still preventing Gibbs oscillations near shocks.

Demonstrations of the accuracy of this code in reproducing physical behavior such as nonlinear steepening, shock wave formation, N wave lengthening, and wavefront focusing and folding are presented in section III.

As described in the companion paper by Lociey et al.,\(^{24} \) atmospheric turbulence is incorporated into the NPE as perturbations in the sound speed \( c'(x, y) \) that arise from temperature fluctuations that follow a modified von Kármán energy spectrum. The contribution of turbulent velocity fluctuations to \( c' \) could be added separately, but it is expected that the wave form distortions produced by including only one random component of \( c' \) are qualitatively indistinguishable from those produced by sound speed perturbations formed from the superposition of two random components that follow the same spectrum.

Because significant sound speed perturbations \( (c'/c_0 > .01) \) can have a length scale as small as one centimeter, it is necessary to make the time step sufficiently small to capture the effect of the full spectrum of turbulence on the evolution of the wave field near the shock front. In other words, because turbulence is moving across the computational domain with speed \( c_0 \), the computational advantage of large time steps
no longer applies. To avoid unstable behavior in the solution, the time step needed to be on the order of
\[ \Delta t = \Delta x/c_0 = 10^{-5} \text{ seconds}. \]

The spatial resolution in the primary direction of propagation, \( \Delta x \), was initially chosen so that the shock profile of the initial steady state N wave is resolved with at least three points, ensuring that fine structures, such as spikes, that might develop later can be also be resolved. The Taylor shock width for a sonic boom with an amplitude of 100 Pa is about 3 cm, so \( \Delta x \) was chosen to be .01 m. However, significant computational efficiency was gained by ensuring that the grid advanced exactly \( \Delta x = c_0\Delta t \) for each time step of \( 10^{-5}s \), so the grid spacing was reduced to .00343 m. This enables the array of \( c' \) values to be calculated only once for the entire computational domain at the start, then incrementally shifted as the domain travels with speed \( c_0 \), with only one new row of \( c' \) values calculated at each time step. Because the transverse component of wave propagation occurs over a longer time scale (as long as the narrow angle approximation is valid), the spatial resolution in the transverse direction, \( \Delta y \) can be larger. This value was chosen to be .1 m.

The size of the computational domain in the \( x \) direction is determined by the length of the sonic boom waveform, the region behind the tail shock where the solution will show scattering and diffraction effects, and the maximum distance the front shock is expected to travel forward relative to the grid because of sound speed perturbations and nonlinear effects. A shock travels with speed

\[ v_{sh} = \frac{\beta P_{sh}}{2\rho c_0} \]  

relative to the grid moving with speed \( c_0 \).

For a shock overpressure of \( P_{sh} = 100 \) Pa and \( c_0 = 343 \) m/s, \( v_{sh} = .15 \) m/s, so the shock front may advance about 50 cm within the grid as it propagates from a height of 1 km to the ground. Over the span of about 100 m, which is the largest length scale of turbulence included in this study, some regions of the shock front may experience an average excess sound speed of up to 4 m/s; over longer distances, the average sound speed perturbation for any region should approach zero. However, within the 100 m span, which is traversed in approximately .3 seconds, the shock may advance up to 1.2 m within the grid. Thus, the total computational buffer that should be included ahead of the shock is about two meters, or 600 grid points (times the number of points in the transverse direction). A typical value for the length of the N wave is 30 m, with an extra 30 m behind the tail shock. The total length of the domain was then about 62 m, which required over 14,000 grid points.

![Diagram indicating the boundary conditions on the computational domain and the relative motion of turbulence.](image)

Figure 1: Diagram indicating the boundary conditions on the computational domain and the relative motion of turbulence.

In the transverse direction, the computational domain should be large enough to encompass the outer length scale of turbulence. Periodic boundary conditions in the transverse direction necessitated creating a mirror image of the sound speed perturbation array, with the axis of symmetry along the \( x \) axis centered between the boundaries. Thus, the transverse dimension needs to be twice the outer length scale of turbulence.
The largest value for the outer turbulence length scale used in the present study was 100 m, which required 2000 grid points along the y axis. The largest runs, then, required $15,000 \times 2,000 = 30$ million grid points.

No boundary conditions are applied to the front and rear of the grid, except that the wave field is assumed to be quiescent ahead of and far behind the shock. Figure 1 illustrates the boundary conditions applied to the NPE.

### III. Benchmark tests

To demonstrate the ability of the modified NPE to correctly model nonlinear behavior in one dimension, a plane sine wave (one cycle) is propagated parallel to the x axis in a homogeneous medium. In this case, the NPE (in the moving reference frame) reduces to

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x}\left(\beta \frac{p^2}{2 \rho c_0} - \delta_{\text{eff}} \frac{\partial p}{\partial x}\right) = 0$$

which is equivalent to the well known Burgers equation. Solutions to this equation are wave functions that steepen as they propagate, until a shock is formed. The shock formation distance, for an initially sinusoid waveform, is given by

$$x_{sh} = \frac{\lambda \rho c^2}{2\pi \beta p_{pk}}$$

Using $\lambda = 1.0\text{m}$ for the wavelength, $p_{pk} = 100\text{Pa}$ for the initial peak amplitude, and the standard values of $\rho$ and $c$ for air at sea level, the shock formation distance is 187 meters, which occurs after 545 seconds of propagation. Beyond this distance, the front shock will propagate with speed $c + \frac{\beta p_{pk}}{2\rho c_0}$, while the tail shock travels with speed $c - \frac{\beta p_{sh}}{2\rho c_0}$ and the zero crossing point of the waveform travels with speed $c$. The front shock is overtaken by the waveform immediately behind it, with the result that an N wave is eventually formed that decreases in amplitude as it propagates.

For this test, the grid spacing was $\Delta x = .005 \text{m}$, resolving the initial pulse with 200 points, and the time step was $\Delta t = .0147\text{s}$. The latter value was chosen to be as large as possible while satisfying the condition of numerical stability $\Delta t < \Delta x/v_{\text{max}}$, where $v_{\text{max}} = \frac{\beta p_{pk}}{\rho c}$ in the reference frame moving with speed $c$. The domain size was made large enough to accommodate movement of the leading and trailing edges of the waveform after shock formation. The dissipation coefficient $\delta_{\text{eff}}$ was set to zero to demonstrate the ability of the algorithm to handle discontinuities.

Figure 2 shows the NPE solution at several times before and after shock formation. In addition to qualitatively correct behavior, the solution exhibits the correct shock formation distance, shock speed, and rate of shock amplitude decay predicted by Whitham’s rule.

A second test of the NPE demonstrates its ability to correctly model the two dimensional linear propagation of a step shock with a curved wavefront. In this case, the NPE is solved with $\beta = 0$, so that only the dissipation and diffraction terms are used. The solution along the x axis that passes through the center of wavefront curvature is compared to an analytical solution derived using time-dependent Green functions. At the shock itself, the amplitude is given by geometric acoustics, but the solution behind the shock depends on the contributions of diffracted waves from neighboring regions of the shock front. The full two dimensional wave numerical solution can also be plotted with high resolution, enabling the user to visualize complicated wave evolution due to inherently multidimensional effects.

The initial condition is a nominally planar step shock with an axisymmetric concave wavefront and a fully developed Taylor shock profile. The curvature of the wavefront is such that the focal point exists 30 meters from the initial shock location. The computational mesh contains 800 points along the x axis (direction of propagation) and 1000 points along the y axis, with mesh spacing $\Delta x = .002\text{m}$ and $\Delta y = .02\text{m}$ that resolves the shock width with about five points. A relatively small time step, $\Delta t = 2c/\Delta x = .000012\text{s}$, was required to ensure numerical stability for the diffraction term in the NPE. The focal point was reached by the wave at approximately $7,500$ time steps.

Examples of the full wave field NPE solution before and shortly after the focal points are reached are shown in figures 3 and 4. The wave is travelling to the right in these images, with wave amplitude shown on the vertical axis. In the vicinity of the wavefront focus, the shock amplitude increases significantly, as expected by geometric acoustics, but the wave field behind the shock, not so easily predicted, is of greater interest.
Figure 2: Nonlinear steepening of an initially sinusoidal pulse in a reference frame moving with the ambient wave speed. The NPE solution at every twelfth time step is shown. Shock formation occurs at time step 37. After this time, both the front and tail shock move relative to the computational domain. The ordinate is normalized amplitude; the abscissa is position in meters.

Figure 3: Full wave field view of linear NPE solution with diffraction for a focusing step shock, prior to reaching the focal point. Propagation is to the right (x axis) and pressure is represented along the z axis (out of the page).

Figure 4: Full wave field view of linear NPE solution with diffraction for a focusing step shock, just beyond the focal point.

A comparison of the NPE solution with the analytical solution for the shock profile along the propagation axis that intersects the focal point is shown in figure 5. Numerical dissipation limits the accuracy of the NPE right at the shock, but the solution behind the shock agrees well with linear theory.

IV. Results with Turbulence

The performance of the NPE on benchmark tests provides confidence that the nonlinear solution with turbulence will give reliable results. Initially planar N waves of length 30 m and width ranging from 10 m to 200 m, with a fully developed Taylor shock profile, are propagated through a turbulent field as described in section II. The width of the computational domain was varied to accommodate different outer length scales of turbulence to investigate their effect on sonic boom distortion.

A sample of results for a relatively small wave field, corresponding to an outer turbulence length scale of 10 m, is shown in figures 6 and 7. Here, the wave has propagated .7 seconds, or approximately 200 m. For this realization of turbulence, the sound speed fluctuations have a magnitude of about 1 m/s. Note the symmetry of the turbulent field, which is required for the use of simple periodic boundary conditions in the transverse
direction. Three shock profiles excerpted from the full wave field are shown on the right, with detailed views in the vicinity of the shocks. The rippled profiles behind the shock are consistent with previous NPE calculations that propagated a shock with multiple wavefront curvature scales in a homogeneous medium.

The next pair of figures, 8 and 9, show the same wave 200 ms later. It is interesting to note both the temporal and spatial variations in the shock profiles, which are consistent with the variations seen in recorded sonic booms during flight tests with turbulence.4 These results demonstrate qualitatively the type of wave distortion that turbulence can produce over propagation distances of a few hundred meters. However, as the wave propagates further, the solution at a given point of the wavefront depends on an increasingly larger region of the initial wave field and the medium it propagates through. In other words, a large scale wavefront ripple formed by a large turbulent eddy will focus at larger distances than small wavefront ripples, but the effect on shock amplitude is likely to be larger. The largest turbulence length scale of interest is limited by requiring that focal distances be less than the height of the turbulent boundary layer.

To capture the waveform distortion effects of any particular turbulence length scale, the computational domain must be sufficiently wide to encompass at least one full spatial period of wave speed perturbations. Preliminary results using the NPE with different outer turbulence scales (and corresponding domain sizes) suggests that peaking and rounding of the sonic boom profile intensifies when the larger perturbation scales are included.

V. Conclusions and Recommendations

The NPE is well suited to simulating sonic boom propagation through individual realizations of atmospheric turbulence, making it possible to study the cumulative effects of turbulence, gain insight into the physical mechanisms of waveform distortion, and to perform monte carlo studies that relate parameters of turbulence strength to probabilities of particular deviations from the upper atmosphere sonic boom.

Currently, only temperature fluctuations are incorporated into the sound speed perturbation, \( c' \) that appears in the NPE. Velocity fluctuations that have a component in the propagation direction also contribute to \( c' \); these can be added by modeling their spatial and energy distribution with a von Kármán spectrum, as is currently done with the temperature.
The computational speed and efficiency of the NPE could be significantly enhanced by employing an adaptive grid size scheme that ensures high resolution at, and just behind, shocks, but allows coarse spatial resolution where the solution is smooth. Ultimately, it is desirable to extend the current two dimensional solution of the NPE to three dimensions, as in equation 1. The multi-dimensional flux limiting approach employed in the present version of the NPE is theoretically well suited to this task and may be more computationally efficient than time step splitting approaches.

References


Figure 6: Full wave field view of the NPE solution at $t = 700$ms for an N wave propagating through turbulence. Turbulence is represented by a perturbation of the local sound speed, shown as colour contours below the wave field (contour scale units are meters per second).

Figure 7: Select wave profiles taken from the full wave field at left, with exploded view near shocks.
Figure 8: Full wave field view of the NPE solution at $t = 900\text{ms}$ for an N wave propagating through turbulence. Turbulence is represented by a perturbation of the local sound speed, shown as color contours below the wave field (contour scale units are meters per second).

Figure 9: Select wave profiles taken from the full wave field at left, with exploded view near shocks.