Heteroclinic and Homoclinic Connections for Two Classes of Hamiltonian Systems

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Abstract

We study a class of Hamiltonian systems, (HS) $\ddot{q}(t) = -V_q(t,q(t))$ under various conditions on the potential $V$. In the first section, we consider the case when $V$ is bounded from above and has a finite collection of non-degenerate absolute maxima with $V(t,\xi) = 0$ for all $\xi$ in the set of maxima of $V$. We consider minimax values found by paths connecting translates of minimizers of an appropriate functional, and show that under a mild condition, there are other solutions besides minimizers. Then, under a non-degeneracy condition, we also establish the existence of infinitely many distinct 2-bump solutions of (HS), each of which is close to a chain of mountain pass type solutions.

Then we turn our attention to the case of a single equation and $V$ has two distinct non-degenerate maxima at different levels: 0 is a local maxima and $\xi \neq 0$ is an absolute maxima. Under general conditions on $V$, we show that there is a solution of (HS) homoclinic to 0. Then, supposing that another condition holds, we show the existence of infinitely many solutions of (HS) homoclinic to 0 that are distinguished from one another by the number of times and regions where the solutions is away from 0. As a corollary, we show that if there is a solution of (HS) to $\xi$, then there are infinitely many solutions of (HS).
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Chapter 1

Introduction

In this thesis, we concern ourselves with finding solutions of the following Hamiltonian system of ordinary differential equations:

\begin{equation}
\ddot{q}(t) = -V_q(t, q(t)),
\end{equation}

where the potential \(V(t, q)\) is periodic in time \(t\), and \(q(t) \in \mathbb{R}^d\) for \(d = 1, 2, \ldots\). Such systems arise as models for a wide variety of physical phenomena. For example, if the potential is also periodic in the space variable \(q\), then such an equation models the behavior of a multiple pendulum. If \(-V\) is a so-called double well potential, then (HS) is a model for phase transition phenomena. These are the two cases considered in this thesis. Because of its common occurrence, (HS) has been studied extensively. An interesting question is the existence of homoclinic or heteroclinic solutions of (HS), that is, solutions \(q(t)\) of (HS) such that \(\lim_{t \to \pm \infty} q(t)\) exists and \(q(-\infty) = \lim_{t \to -\infty} q(t) = \lim_{t \to \infty} q(t) = q(\infty)\) or \(q(-\infty) \neq q(\infty)\). Such questions have been studied extensively, for example by perturbation methods using a Melnikov function (as in [10]) and shooting methods. Prior to the late 1980’s and early 1990’s, a number of papers using variational methods to study homoclinics and heteroclinics appeared in the Russian literature, for example [2] and [3], as well as the survey article [13] and the references contained [13]. In the late 1980’s and early 1990’s, starting with the papers of Coti Zelati, Ekeland and Séré
in [6], Séré in [21] and Rabinowitz in [18], variational methods were again used to investigate the existence of homoclinic and heteroclinic solutions of (HS). In several papers, see e.g. [15], Mather has used variational methods in a different spirit to address homoclinic and heteroclinic solutions of (HS).

We will use variational methods here. We consider a functional

$$I(u) := \int_{\mathbb{R}} \left( \frac{1}{2} |\dot{u}(t)|^2 - V(t, u(t)) \right) dt.$$  

Under general conditions on the potential $V$, critical points of $I$ correspond to solutions of (HS). Our results fall into two categories:

1. The potential $V$ has a finite number of points $K(V) = \{0, \xi_1, \ldots, \xi_k\}$ such that $V(t, q) < V(t, \xi_i) = 0$ for all $q \not\in K(V)$, and

2. The potential has a pair of isolated local maxima, say at $q = 0, q = 1$, and $V(t, 0) < V(t, 1)$.

Let us consider Case (1). We assume that

(V1) $V \in C^2(\mathbb{R} \times \mathbb{R}^d, \mathbb{R})$, and $V$ is 1-periodic in $t$.

(V2) There is a finite set of points $K(V) = \{0, \xi_1, \xi_2, \xi_3, \ldots, \xi_k\}$ such that $V(t, \xi_i) = 0 > V(t, x)$ for $x \not\in K(V)$ and all $\xi_i \in K(V)$.

(V3) $\liminf_{|x| \to \infty} V(t, x) \leq -\alpha < 0$, uniformly in $t$.

(V4) $V_{qq}(t, \xi_i)$ is negative definite for $i = 1, 2, \ldots, k$, uniformly in $t$.

Much more is known in this case. This is partly due to the fact that heteroclinic solutions arise when one minimizes $I$ over the set $\Lambda$ of $W^{1,2}_\text{loc}(\mathbb{R})$ functions such
that \( q(-\infty) = 0, q(\infty) \neq 0 \). For example, in [19], Rabinowitz showed that when \( K(V) = \{0, \xi\} \), there exist heteroclinic solutions of (HS) connecting 0 to \( \xi \) and \( \xi \) to 0. In addition, it was shown in [19] that if solutions of (HS) are isolated, then there are infinitely many heteroclinic and homoclinic solutions of (HS), each shadowing a “chain” of minimizers, that is a collection \( \{v_1, v_2, \ldots, v_k\} \) of solutions of (HS) with \( v_n(\infty) = v_{n+1}(-\infty) \), and each \( v_i \) is a minimizer of \( I \) over an appropriate class of functions. In [19], an indirect variational argument involving estimates on \( I' \) and deformations is used. More recently, in [20], these indirect arguments have been replaced by more direct constrained minimization arguments, in which a solution is found subject to the constraint that admissible functions are close to an approximation of the chain \( \{v_1, v_2, \ldots, v_k\} \).

In [22], Strobel considered the case of when \( V \) is periodic in all of its arguments. In [22], the set of global maxima of \( V \) consists of a lattice of points in \( \mathbb{R}^d \), which after re-scaling can be assumed to be \( \mathbb{Z}^d \). In [22], it was shown that given any \( \beta \in \mathbb{Z}^d \) and \( \eta \in \mathbb{Z}^d \) with \( \eta \neq \beta \), there is a heteroclinic solution \( q \) of (HS) such that \( q(-\infty) = \eta \) and \( q(\infty) \neq \eta \). Moreover, given any \( \eta \neq \beta \), there is a chain \( \{v_1, v_2, \ldots, v_k\} \) of minimizing solutions such that \( v_1(-\infty) = \eta \) and \( v_k(\infty) = \beta \). Then, assuming these minimizing solutions of (HS) are isolated and using an indirect argument similar to that of [19], Strobel showed that there is in fact a solution \( q_{\eta, \beta} \) of (HS) such that \( q_{\eta, \beta}(-\infty) = \eta, q_{\eta, \beta}(\infty) = \beta \) and \( q_{\eta, \beta} \) shadows a chain of minimal solutions of (HS).

More recently, in [20], Rabinowitz reproved Strobel’s result about the existence of \( q_{\eta, \beta} \) using more direct minimization arguments. These results provide a variational analogue of gluing methods in dynamical systems based on the use of a Melnikov function.
Let us assume for the moment that $K(V) = \{0, \xi\}$. Because of the periodicity in time of the potential $V$, if $q(t)$ is a solution of (HS), $\tau_k q(t) := q(t - k)$ is also a solution for any $k \in \mathbb{Z}$. Thus, if $q(t)$ is a minimizer of $I$, then $\tau_k q(t)$ is also a minimizer. If minimizers of $I$ are isolated, then an interesting question is the existence of solutions of (HS) of mountain pass type. To this end, for $q$ a minimizer of $I$ over the set of $W^{1,2}_{loc}(\mathbb{R})$ functions that connect 0 to some $\xi \in K(V)$, $\xi \not\equiv 0$, we introduce the functional\[ \tilde{J}_q(u) := I(u + q) - I(q) \]for $u \in W^{1,2}(\mathbb{R})$. Then, 0 and $v(t) = \tau_1 q(t) - q(t)$ are both minimizers of $\tilde{J}_q$. If the minimizer $q$ is isolated from other minimizers of $I$, then\[ c := \inf_{h \in \Gamma} \max_{s \in [0,1]} \tilde{J}_q(h(s)) > 0 \]where\[ \Gamma := \{h \in C([0,1], W^{1,2}(\mathbb{R})) \mid h(0) \equiv 0, h(1) \equiv \tau_1 q - q\}. \]

We then have the following theorem:

**Theorem 1.0.1.** If $c$ is small, then there is a $u \in E$ such that $\tilde{J}_1(u) = c$ and $\tilde{J}_1(u) = 0$.

Thus, $u + q$ solves (HS) and is not a translate of $q$. If $c$ is not small, then we still have the following:

**Theorem 1.0.2.** Let $q$ be a minimal heteroclinic solution of (HS), connecting 0 to $\xi$, and $p$ be a minimal heteroclinic solution of (HS), connecting $\xi$ to 0. If $c \neq k_1 I(q) + k_2 I(p)$ for $k_1, k_2 \in \mathbb{N}$, $k_1, k_2 \geq 0$ then there is a non-constant $v$ with $I(v) < \infty$, $I'(v) = 0$, $v(\pm \infty) \in K(V)$ and $v \not\equiv p, v \not\equiv q$. 
Both of these theorems are proved after we have obtained detailed information about the structure of Palais-Smale sequences of the functional $I$ (that is, sequences $u_n$ such that $I(u_n)$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$).

Theorems 1.0.1 and 1.0.2 hold when $K(V)$ is finite. Consider the case when $V$ is periodic in all of its arguments, i.e. $K(V) = \mathbb{Z}^d$, as in [22]. Because of the useful $L^\infty$ bounds from [19], [22], we can modify $V$ to get a potential $\tilde{V}$ with $K(\tilde{V})$ finite, and then apply the previous results to get corresponding existence results for this setting.

Once we know something about the existence of solutions of (HS) of mountain pass type, we can move on to ask whether or not there are solutions $\tilde{v}$ of (HS) that “shadow” chains $\{\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_k\}$ where each $\tilde{v}_i$ is a solution of (HS) of mountain pass type. To answer this question, we need to make an assumption about the solutions of (HS). As is customary for such situations, we need a sort of non-degeneracy condition. A suitable condition is that the solutions of (HS) are isolated, in the sense that there is an $r > 0$ such that if $w_1, w_2$ are distinct solutions of (HS), then $\|w_1 - w_2\|_{W^{1,2}(\mathbb{R})} \geq r$. Moreover, we need to recall the indirect methods of [19] and [22]. Because we are looking at shadowing mountain pass type critical points rather than minimizers, the argument we present is much more complicated and technical than that of either [19] or [22]. Our argument is somewhat similar to the work of Coti Zelati and Rabinowitz in [8], where it is shown how to “glue” solutions of mountain pass type. However, in [8], Coti Zelati and Rabinowitz assume that the potential $V$ has an isolated equilibrium, and so if $I(q) < \infty$, then $q(-\infty) = q(\infty) = 0$. In our case, $I(q) < \infty$ implies only that $q(\pm \infty) \in K(V)$. This increases the difficulty greatly. Let $q_1, q_2$ be the minimizers...
of $I$ connecting 0 to $\xi$ and $\xi$ to 0, respectively. Then, let

$$\hat{c}_i := \inf_{h \in \Gamma_i} \max_{s \in [0,1]} J_{q_i}(h(s))$$

where

$$\Gamma_i := \{ h \in C([0,1], W^{1,2}(\mathbb{R})) \mid h(0) \equiv 0, h(1) \equiv \tau_1 q_i - q_i \}.$$ 

We prove the following theorem:

**Theorem 1.0.3.** If $\hat{c}_1, \hat{c}_2$ are both sufficiently small, and critical points of $J_{q_i}$ with values close to $\hat{c}_i$ are isolated, then are infinitely many solutions $\hat{v}$ of (HS) homoclinic to 0 such that $\hat{v}$ is “close” to the chain $(u_1 + q_1, u_2 + q_2)$, where $u_i$ is a critical point of $J_{q_i}$ with critical value $\hat{c}_i$ (so $u_i + q_i$ is a solution of (HS) heteroclinic from 0 to $\xi$ ($i = 1$) and heteroclinic from $\xi$ to 0 ($i = 2$)).

The solutions found are distinguished by the amount of time that they spend close to $\xi$. Notice that $I(\hat{v})$ will be close to $\hat{c}_1 + I(q_1) + \hat{c}_2 + I(q_2)$, and as the amount of time that $\hat{v}$ spends close to $\xi$ increases, the corresponding critical value of $I$ will get closer and closer to $\hat{c}_1 + I(q_1) + \hat{c}_2 + I(q_2)$. Rabinowitz’ result in [19] implies that there are critical values of $I$ that accumulate at $I(q_1) + I(q_2)$, and we prove the same is true at $\hat{c}_1 + I(q_1) + \hat{c}_2 + I(q_2)$. It seems likely that there are other possible accumulation points of critical values of $I$ corresponding to other types of solutions. For example, if there are solutions $\tilde{v}_1, \tilde{v}_2$ of (HS) heteroclinic from 0 to $\xi$ and vice versa, we would expect there to be critical values of $I$ accumulating at $I(\tilde{v}_1) + I(\tilde{v}_2)$, and so on. Notice that because of the isolation assumptions, none of these results apply to the case when $V$ is independent of time.

We turn now to Case (2). Here, we assume that $d = 1$. We make the following assumptions about $V$:
(DV1) $V \in C^3(\mathbb{R} \times \mathbb{R}, \mathbb{R})$

(DV2) $V$ is 1 periodic in $t$: $V(t + 1, q) = V(t, q)$

(DV3) $q = 0$ and $q = 1$ are non-degenerate local maxima of $V$ for every $t$. Moreover, we assume $V(t, 0) = 0$ and there exists a constant $c_1 > 0$ such that $V(t, 1) \geq c_1$ for all $t$.

(DV4) $q = \frac{1}{2}$ is a non-degenerate local minimum of $V$, and there is a constant $c_2 < 0$ such that for every $t$, $V(t, \frac{1}{2}) \leq c_2$.

(DV5) $V_q(t, x) \neq 0$ for $x \in (1/2, 1)$, and there is a constant $\Lambda > 0$ such that $V(t, x) \leq -\Lambda x^2$ for $0 \leq x \leq 1/2$.

(DV6) There are constants $\alpha_0, \beta > 0$ such that $V(t, q) < -\alpha_0 q^2 + \beta$.

(DV7) $V_{qq}, V_{tq}$ and $V_{qqq}$ are bounded.

(DV1) is a technical assumption. Because of the methods we use to study Case (2), the conditions (DV6-7) are not essential constraints. Notice that because $V(t, 0) \equiv 0$ and $V(t, 1) < 0$, the functional $I$ is not bounded from below on $W^{1,2}(\mathbb{R})$. Thus, we cannot minimize, and the techniques of Case (1) do not apply. In particular, it is unclear as to how to get a nice description of the splitting of (PS) sequences. However, if we look at $I$ restricted to periodic functions, then $I$ is bounded from below - but the minimizer will be the constant solution $q \equiv 1$. Thus, we cannot use the same approach as in Case (1). It turns out that for this setting (HS) admits homoclinic solutions. We prove
Theorem 1.0.4. If $V$ satisfies (DV1-7), then there is a solution $q$ of (HS) homoclinic to 0 such that there are exactly two points $a, b$ where $q(a) = 1/2 = q(b)$.

We know of only two results in Case (2) which do not use perturbation arguments. In [16], Petroll considers a surface $M$ diffeomorphic to the cylinder $S^1 \times [0, 1]$ for which both ends can be parametrized by geodesics, $\gamma_0, \gamma_1$, and such that $\gamma_0$ is a local minimizer of the length functional and $L(\gamma_1) < L(\gamma_0)$. Under these conditions, Petroll proves the existence of a geodesic homoclinic to $\gamma_0$. In Petroll’s proof, the homoclinic solution there is found as a limit of a sequence of periodic geodesics. In [16], the key is to use curvature flow: it decreases the length of curves and simplifies the geometry of the curves in question. We give two proofs of our result. In the first method, we use an appropriate semi-linear heat flow:

\[
(PDE) \quad w_s(s, t) = w_{tt}(s, t) + V_q(t, w(s, t)) \quad w(0, t) = u(t).
\]

Then, an appropriate choice of an initial curve and the flow are used to construct the subharmonic periodic solutions $q_k$ to the Hamiltonian system. The key step in the proof is the existence of a subsequence $q_{k_j}$ of subharmonics for which the amount of time that each $q_{k_j}$ spends larger than 1/2 in any given period is bounded independently of $j$. This in turn relies on a very useful fact about the solutions of scalar nonlinear parabolic partial differential equations like (PDE): the number of zeros of a solution $w(s, t)$ of (PDE) is non-increasing as $s$ increases, as Angenent showed in [1]. For example, in [16], Petroll uses the fact that the curve shortening flow decreases the number of interesections with any given geodesic, which is a variant on this same principle. The second method uses a variant of the mountain pass lemma (with the standard deformation replaced by the heat flow arising from
(PDE)) to get a sequence of critical values corresponding to subharmonic solutions of (HS). Then, an argument involving the maximum principle is used to show that there is a subsequence for which the amount of time that any one spends larger than 1/2 is bounded.

In [7], Coti Zelati and Rabinowitz use a minimization argument to show the existence of solutions of (HS) heteroclinic from 0 to 1. Then, by a gluing procedure, they prove the existence of infinitely many heteroclinics and homoclinics. They consider “slowly” varying potentials $V(t, q) = a(t)\tilde{V}(q)$, where the periodic term $a(t)$ has a long period, and $\sup a(t) - \inf a(t)$ is small. Thus, in a sense, [7] is a perturbation result. In contrast, we do not make such an assumption about the form of the potential.

In addition to Theorem 1.0.4, we show that if $V$ satisfies another condition, then (HS) possesses infinitely many multi-bump like solutions, that is solutions of (HS) homoclinic to 0 that have disjoint intervals of time where they are far from 0. Thus, the graphs of such solutions have multiple bumps, where the solution remains away from 0. Our proof relies on a very geometrical construction of such solutions.
Chapter 2

Splitting of Palais-Smale Sequences

2.1 Preliminaries

In this section, we investigate the existence of heteroclinic and homoclinic solutions of the Hamiltonian system

\[(HS) \quad \ddot{q}(t) = -V_q(t, q(t)),\]

where \(V\) is a potential with finitely many wells, and \(q(t) \in \mathbb{R}^d\). Throughout, \(\| \cdot \|_{L^p}\) and \(\| \cdot \|_{W^{1,2}} = \| \cdot \|_E\) will always be the standard norms on \(L^p(\mathbb{R}), W^{1,2}(\mathbb{R}) := E\) respectively, while \(\| \cdot \|_{L^p(A)}, \| \cdot \|_{W^{1,2}(A)}\) will be the standard norms on \(L^p(A), W^{1,2}(A)\) for \(A \subset \mathbb{R}\). Next, \(\langle \cdot, \cdot \rangle_E\) will be the inner product in \(E\). Finally, \(B_r(y)\) will denote the ball of radius \(r\) centered at \(y\). From context, it will be clear what space \(B_r(y)\) is in. For example, if \(u \in E, B_r(u) \subset E\), while if \(\xi \in \mathbb{R}^d\), then \(B_r(\xi) \subset \mathbb{R}^d\).

We look for solutions of \((HS)\) as critical points of the functional

\[I(q) := \int_{\mathbb{R}} \left( \frac{1}{2} |\dot{q}(t)|^2 - V(t, q(t)) \right) dt\]

for \(q \in \dot{E} := W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^d)\). We make \(\dot{E}\) into a Hilbert space by using the inner
product
\[ \langle u, v \rangle_E := \langle u(0), v(0) \rangle + \int_{\mathbb{R}} \langle \dot{u}(t), \dot{v}(t) \rangle dt, \]
where \( \langle \cdot, \cdot \rangle \) is the standard inner product in \( \mathbb{R}^d \). Moreover, we make the following assumptions on the potential \( V \):

(V1) \( V \in C^2(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}) \), and \( V \) is 1-periodic in \( t \).

(V2) There is a finite set of points \( K(V) = \{ \xi_1, \xi_2, \xi_3, \ldots \xi_k \} \) such that \( V(t, \xi_i) = 0 > V(t, x) \) for \( x \not\in K(V) \) and all \( \xi_i \in K(V) \).

(V3) \( \liminf_{|x| \to \infty} V(t, x) \leq -\alpha < 0 \), uniformly in \( t \).

(V4) \( V_{q_i}(t, \xi) \) is negative definite for \( i = 1, 2, \ldots k \).

We may assume without loss of generality that \( \xi_1 = 0 \).

Remark 2.1.1. Notice that (V2) and (V4) imply that there are constants \( 0 < \beta_1 < \beta_2 \) and \( \eta > 0 \) such that if \( x \in B_\eta(\xi) \), then \( \beta_1|x - \xi|^2 \leq -V(t, x) \leq \beta_2|x - \xi|^2 \).

Definition 2.1.2. For any \( k \in \mathbb{Z} \), \( v \in \mathring{E} \), we define \( \tau_k v \) by \( \tau_k v(t) := v(t - k) \).

Notice that (V1) implies that \( I(\tau_k v) = I(v) \). Thus, \( I \) is invariant under a \( \mathbb{Z} \) action. Notice that this also implies that \( I \) does not satisfy the (PS) condition. Suppose that \( K(V) = \{ 0, \xi \} \). In this case, it is known (see [19]) that there exist heteroclinic solutions \( q_1, q_2 \) of (HS) connecting 0 to \( \xi \) and \( \xi \) to 0. These solutions are found by minimizing \( I \) over an appropriate class of functions: \( q_1 \) is a minimizer of \( I \) over the class of functions in \( \mathring{E} \) starting at 0 when \( t = -\infty \), and ending up at \( \xi \) when \( t = +\infty \). Similarly, \( q_2 \) is the minimizer of \( I \) over the set of functions that start at \( \xi \) and end at 0. In addition to these solutions of (HS), Rabinowitz also showed in
[19] that given any chain of minimizers, there is an actual solution of (HS) that is “close” to the chain in an appropriate sense, provided a non-degeneracy condition holds, namely that the solutions are isolated. Recently, it has been shown in [20] that these “shadowing” solutions can be found by a minimization argument. In addition, in [22], Strobel considered the case where $V$ is periodic in all of its arguments. Thus, the set $K(V)$ is a lattice of points. We may assume that in this case, $K(V) = \mathbb{Z}^d$. We generalize our results to consider this case. In this section, we want to find heteroclinic solutions of (HS) that are not minimizers. To do this, we will introduce a minimax value $c$, as the minimax of $I$ over all paths in $\hat{E}$ that connect a minimizer $q$ of $I$ and its translate $\tau_1 q$. Standard theorems (see [25]) then imply the existence of a Palais-Smale (PS) sequence $v_n$ such that $I(v_n) \to c$ as $n \to \infty$. Since $I$ does not satisfy the (PS) condition, it is necessary to investigate how such sequences “split” into convergence sequences. This is the goal of this section. Once this is done, we can use our knowledge of how (PS) sequences split to show the existence of solutions of (HS) that are not minimizers.

For future reference, we note the following

**Lemma 2.1.3.** [19], [22]

(i) For every $M$, there is a $C(M)$ such that if $I(q) < M$, then $\|q\|_{L^\infty} < C(M)$. 

(ii) If $I(q) < \infty$, then $\lim_{t \to -\infty} q(t) := q(\infty)$ and $\lim_{t \to -\infty} q(t) := q(-\infty)$ exist, and $q(\pm \infty) \in K(V)$. 

We have as a consequence of (V4) the following useful lemma:

**Lemma 2.1.4.** If $I(q) < \infty$, then $q - q(-\infty) \in W^{1,2}(-\infty, 0)$ and $q - q(\infty) \in W^{1,2}(0, \infty)$. 

Proof. It suffices to show that \( q - q(\infty) \in L^2([0, \infty)) \). Pick \( T \) so large that for \( t > T, |q(t) - q(\infty)| < \eta \). Then, by Remark 2.1.1 above and Lemma 2.1.3, we have
\[
\infty > I(q) \geq \int_T^\infty -V(t, q(t))dt \geq \int_T^\infty \beta_1|q(t) - q(\infty)|^2dt,
\]
and the lemma follows. \( \square \)

Thus, if \( q_1, q_2 \in \hat{E} \) are two functions such that \( I(q_1), I(q_2) < \infty \), \( q_1(-\infty) = q_2(-\infty) \) and \( q_1(\infty) = q_2(\infty) \), then \( q_1 - q_2 \in E \).

**Definition 2.1.5.** For any \( \chi \in \hat{E} \) such that \( I(\chi) < \infty \), we define
\[
J_\chi(u) := I(\chi + u) \text{ for any } u \in E.
\]

**Proposition 2.1.6.** (i) \( J_\chi \in C^1(E, \mathbb{R}) \), and
\[
J'_\chi(u)\varphi = \int_\mathbb{R} \langle \dot{\chi} + \dot{u}, \dot{\varphi} \rangle - \langle V_q(t, \chi + u), \varphi \rangle dt.
\]
(ii) \( J'_\chi = D \circ Id + K \), where \( D : E \to E' \) is the duality map, \( Id \) is the identity map on \( E \), and \( K : E \to E' \) is such that if \( \|u_n - u\|_{L^2} \to 0 \) and \( \|u_n\|_{L^\infty} \) is bounded, then \( K(u_n) \to K(u) \) in \( E' \).

With this in mind, we put \( I'(v) := J'_\chi(0) \), so
\[
I'(v)\varphi = \int_\mathbb{R} \langle \dot{v}, \dot{\varphi} \rangle - \langle V_q(t, v), \varphi \rangle dt.
\]
Thus, if \( I(\chi) < \infty \), we have
\[
I'(u + \chi)\varphi = \int_\mathbb{R} \langle \dot{\chi} + \dot{u}, \dot{\varphi} \rangle - \langle V_q(t, \chi + u), \varphi \rangle dt.
= J'_\chi(u)\varphi
\]
for all \( \varphi \in E \). Thus, \( J'_\chi(u) = I'(u + \chi) \), and we can then talk about the convergence of \( I'(v_n) \) as elements of \( E' \). Moreover, if \( I'(v) = 0 \), then \( v \) solves (HS).
Proof. We need to show that

\[ J'_\chi(u)\varphi = \int_\mathbb{R} \langle \dot{\chi} + \dot{u}, \dot{\varphi} \rangle - \langle V_q(t, \chi + u), \varphi \rangle dt. \quad (2.1) \]

Thus, we have

\[ J_{\chi}(u + \varphi) - J_{\chi}(u) - J'_\chi(u)\varphi = \int_\mathbb{R} \frac{1}{2} |\dot{\chi} + \dot{u} + \dot{\varphi}|^2 - V(t, \chi + u + \varphi) dt \]

\[ - \int_\mathbb{R} \frac{1}{2} |\dot{\chi} + \dot{u}|^2 - V(t, \chi + u) - J'_\chi(u)\varphi dt \]

\[ = \int_\mathbb{R} \frac{1}{2} \left( |\dot{\chi} + \dot{u} + \dot{\varphi}|^2 - |\dot{\chi} + \dot{u}|^2 - 2\langle \dot{\chi} + \dot{u}, \dot{\varphi} \rangle \right) dt \quad (2.2) \]

\[ - \int_\mathbb{R} (V(t, \chi + u + \varphi) - V(t, \chi + u) - \langle V_q(t, \chi + u), \varphi \rangle) dt \]

\[ = \int_\mathbb{R} \frac{1}{2} |\dot{\varphi}|^2 - \left( V(t, \chi + u + \varphi) - V(t, \chi + u) - \langle V_q(t, \chi + u), \varphi \rangle \right) dt \]

Let us turn our attention to the second term in (2.2):

\[ \int_\mathbb{R} \left( V(t, \chi + u + \varphi) - V(t, \chi + u) - \langle V_q(t, \chi + u), \varphi \rangle \right) dt \]

Because \( V \in C^2 \), we have

\[ V(t, \chi(t) + u(t) + \varphi(t)) - V(t, \chi(t) + u(t)) = \int_0^1 \frac{d}{ds} V(t, u(t) + \chi(t) + s\varphi(t)) ds \]

\[ = \int_0^1 \langle V_q(t, u(t) + \chi(t) + s\varphi(t)), \varphi(t) \rangle ds, \quad (2.3) \]

and so we have

\[ V(t, u(t) + \chi(t) + \varphi(t)) - V(t, u(t) + \chi(t)) - \langle V_q(t, u(t) + \chi(t)), \varphi(t) \rangle \]

\[ = \int_0^1 \langle V_q(t, u(t) + \chi(t) + s\varphi(t)) - V_q(t, u(t) + \chi(t)), \varphi(t) \rangle ds. \quad (2.4) \]
In a similar fashion, we have that

\[
V_q(t, u(t) + \chi(t) + s\varphi(t)) - V_q(t, u(t) + \chi(t))
\]

\[
= \int_0^1 \frac{d}{d\alpha} (V_q(t, u(t) + \chi(t) + s\alpha\varphi(t))) d\alpha
\]

\[
= \int_0^1 V_{qq}(t, u(t) + \chi(t) + s\alpha\varphi(t)) s\varphi(t) \, d\alpha.
\]  

(2.5)

Hence

\[
V(t, u(t) + \chi(t) + \varphi(t)) - V(t, u(t) + \chi(t)) - \langle V_q(t, u(t) + \chi(t)), \varphi(t) \rangle
\]

\[
= \int_0^1 \langle V_q(t, u(t) + \chi(t) + s\varphi(t)) - V_q(t, u(t) + \chi(t)), \varphi(t) \rangle ds
\]

\[
= \int_0^1 \int_0^1 V_{qq}(t, u(t) + \chi(t) + s\alpha\varphi(t)) s\varphi(t) \, d\alpha, \varphi(t) \rangle ds
\]

(2.6)

\[
= \int_0^1 \int_0^1 \langle V_{qq}(t, u(t) + \chi(t) + s\alpha\varphi(t), \varphi(t) \rangle d\alpha ds.
\]

But, then we have that

\[
\left| V(t, \chi(t) + u(t) + \varphi(t)) - V(t, \chi(t) + u(t)) - \langle V_q(t, \chi(t) + u(t)), \varphi(t) \rangle \right|
\]

\[
= \left| \int_0^1 \int_0^1 V_{qq}(t, u(t) + \chi(t) + \alpha s\varphi(t)) s\varphi(t), \varphi(t) \rangle d\alpha d\varphi(t) \right|
\]

(2.7)

\[
\leq \int_0^1 \int_0^1 |V_{qq}(t, u(t) + \chi(t) + \alpha s\varphi(t))||\varphi(t)|^2 d\alpha d\varphi(t).
\]

Now, notice that

\[
|u(t) + \chi(t) + s\alpha\varphi(t)| \leq \|u + \chi\|_{L^\infty} + \|\varphi\|_{L^\infty} \leq \|u + \chi\|_{L^\infty} + 1
\]

if \(\|\varphi\|_{W^{1,2}}\) is small enough. (Here, we make use of the fact that there is a constant \(C\) such that \(\|\varphi\|_{L^\infty} \leq C\|\varphi\|_{W^{1,2}}\).) Thus, for all small enough \(\varphi\), we have

\[
|V_{qq}(t, u(t) + \chi(t) + s\alpha\varphi(t))| \leq \max_{0 \leq t \leq 1, |x| \leq 1 + \|\chi + u\|_{L^\infty}} |V_{qq}(t, x)| =: M(\chi + u).
\]
Notice that $M$ is independent of $\varphi$, supposing that $\|\varphi\|_{W^{1,2}}$ is sufficiently small. Thus, we have

$$
\int_0^1 \int_0^1 \|\dot{V}(t, u(t) + \chi(t) + s\alpha \varphi(t))\|_{op}|\varphi(t)|^2 \, dt \, ds \leq M|\varphi(t)|^2.
$$

Combining all the terms, we then have that

$$
|J_{\chi}(u + \varphi) - J_{\chi}(u) - J'_{\chi}(u)\varphi|
\leq \int_\mathbb{R} \frac{1}{2} |\dot{\varphi}|^2 \, dt + \left| \int_\mathbb{R} \left( V(t, \chi + u + \varphi) - V(t, \chi + u) - \langle V_q(t, \chi + u), \varphi \rangle \right) \, dt \right|
\leq \int_\mathbb{R} \left( \frac{1}{2} |\dot{\varphi}|^2 + M|\varphi|^2 \right) \, dt
\leq \left( \frac{1}{2} + M \right) \|\varphi\|^2_{W^{1,2}}
$$

which implies that $J_{\chi}$ is differentiable, and the derivative is what we have claimed it to be in (2.1). Next, we need to show that $J'_{\chi}$ is continuous. We have

$$
\|J'_{\chi}(u) - J'_{\chi}(w)\| := \sup_{\|\varphi\|_{W^{1,2}} \leq 1} |J'_{\chi}(u)\varphi - J'_{\chi}(w)\varphi|.
$$

But,

$$
\left( J'_{\chi}(u) - J'_{\chi}(w) \right) \varphi = \int_\mathbb{R} \langle \dot{\chi} + \dot{u}, \varphi \rangle - \langle V_q(t, \chi + u), \varphi \rangle \, dt
- \int_\mathbb{R} \langle \dot{\chi} + \dot{w}, \varphi \rangle - \langle V_q(t, \chi + w), \varphi \rangle \, dt
= \int_\mathbb{R} \langle \dot{u} - \dot{w}, \varphi \rangle - \left( \langle V_q(t, \chi + u) - V_q(t, \chi + w), \varphi \rangle \right) \, dt
$$

Now,

$$
\int_\mathbb{R} \langle \dot{u} - \dot{w}, \varphi \rangle \, dt \leq \|\dot{u} - \dot{w}\|_{L^2} \|\varphi\|_{L^2} \leq \|\dot{u} - \dot{w}\|_{L^2},
$$
since $\|\varphi\|_{W^{1,2}} \leq 1$. We also have

\[
V_q(t, \chi + u) - V_q(t, \chi + w) = \int_0^1 \frac{d}{ds} (V_q(t, \chi + w + s(u-w))) ds \\
= \int_0^1 V_{qq}(t, \chi + w + s(u-w))(u-w) ds,
\]

(2.12)

hence

\[
\left| \int_{\mathbb{R}} \langle V_q(t, \chi + u) - V_q(t, \chi + w), \varphi \rangle dt \right| \\
= \left| \int_{\mathbb{R}} \left( \int_0^1 V_{qq}(t, \chi + w + s(u-w))(u-w) ds \right) \varphi dt \right| \\
\leq \int_{\mathbb{R}} \left( \int_0^1 |V_{qq}(t, \chi + w + s(u-w))(u-w)| ds \right) |\varphi| dt \\
\leq \int_{\mathbb{R}} \left( \int_0^1 |V_{qq}(t, \chi + w + s(u-w))|_{op} |u-w| ds \right) |\varphi| dt \\
\leq \int_{\mathbb{R}} \left( \int_0^1 M |u-w| ds \right) |\varphi| dt
\]

(2.13)

But, we have

\[
|\chi(t) + w(t) + s(u(t) - w(t))| \leq \|\chi + u\|_{L^\infty} + \|u - w\|_{L^\infty}.
\]

Hence

\[
|\chi(t) + w(t) + s(u(t) - w(t))| \leq \|\chi + u\|_{L^\infty} + 1
\]

for all $t \in \mathbb{R}$, provided $\|u - w\|_{L^\infty} < 1$. Therefore, for all $s \in [0, 1]$

\[
|V_{qq}(t, \chi + w + s(u-w))| \leq \max_{0 \leq t \leq 1, |x| \leq \|\chi + w\|_{L^\infty}} |V_{qq}(t, x)| =: M
\]

(2.14)

and so by (2.13) and (2.14)

\[
\left| \int_{\mathbb{R}} \langle V_q(t, \chi + u) - V_q(t, \chi + w), \varphi \rangle dt \right| \\
\leq \int_{\mathbb{R}} \left( \int_0^1 M |u-w| ds \right) |\varphi| dt \\
\leq M \|u-w\|_{L^2} \|\varphi\|_{L^2} \\
\leq M \|u-w\|_{L^2}
\]

(2.15)
since \( \| \varphi \|_{W^{1,2}} \leq 1 \). Combining (2.11) and (2.15), we have
\[
|(J'_\chi(u) - J'_\chi(w))\varphi| \leq \| \dot{u} - \dot{w} \|_{L^2} + M \| u - w \|_{L^2},
\]
where \( M \) depends on the \( L^\infty \) norm of \( u - w \). This then implies
\[
\|(J'_\chi(u) - J'_\chi(w))\|_{E'} \leq C \| u - w \|_{W^{1,2}},
\]
so \( J'_\chi \) is locally Lipschitz. Hence \( J_\chi \) is \( C^1 \), which proves part (i).

For part (ii), notice that
\[
J'_\chi(u)\varphi = \int_\mathbb{R} \left( (\dot{u} + \dot{\chi}, \varphi) - \langle V_q(t, \chi + u), \varphi \rangle \right) dt
= \int_\mathbb{R} \left( (\dot{u}, \varphi) + \langle u, \varphi \rangle \right) dt - \int_\mathbb{R} \langle V_q(t, \chi + u), \varphi \rangle dt
= \langle u, \varphi \rangle_{E'} + (-1) \int_\mathbb{R} \langle V_q(t, \chi + u), \varphi \rangle dt
= \langle u, \varphi \rangle_{E'} + K(u)\varphi,
\]
where
\[
K(u)\varphi := \int_\mathbb{R} \langle V_q(t, \chi + u), \varphi \rangle + \langle u, \varphi \rangle - \langle \dot{\chi}, \varphi \rangle dt.
\]
Thus, \( J'_\chi(\cdot) = D \circ Id + K(\cdot) \). To prove (ii), suppose now that \( u_n \to u \) in \( L^2(\mathbb{R}) \) and \( u_n \) is bounded in \( L^\infty(\mathbb{R}) \). Then:
\[
(K(u_n) - K(u))\varphi = \int_\mathbb{R} \left( \langle V_q(t, \chi + u_n) - V_q(t, \chi + u), \varphi \rangle + \langle u_n - u, \varphi \rangle \right) dt,
\]
so if \( \| \varphi \|_{W^{1,2}} \leq 1 \), we have:
\[
\left| (K(u_n) - K(u))\varphi \right| \leq \| u_n - u \|_{L^2} \| \varphi \|_{L^2} + \int_\mathbb{R} \langle V_q(t, \chi + u_n) - V_q(t, \chi + u), \varphi \rangle dt
\leq \| u_n - u \|_{L^2} + \int_\mathbb{R} \langle V_q(t, \chi + u_n) - V_q(t, \chi + u), \varphi \rangle dt.
\]
(2.19)
Now, as usual, we have

\[
V_q(t, \chi + u) - V_q(t, \chi + w) = \int_0^1 \frac{d}{ds}(V_q(t, \chi + u + s(u_n - u))) ds \tag{2.20}
\]

\[
= \int_0^1 V_{qq}(t, \chi + u + s(u_n - u))(u_n - u) ds.
\]

Hence

\[
\left| \int_\mathbb{R} \langle V_q(t, \chi + u_n) - V_q(t, \chi + u), \varphi \rangle dt \right|
\]

\[
= \left| \int_\mathbb{R} \left( \int_0^1 V_{qq}(t, \chi + u + s(u_n - u))(u_n - u) ds, \varphi \right) dt \right|
\]

\[
\leq \int_\mathbb{R} \left| \int_0^1 V_{qq}(t, \chi + u + s(u_n - u))(u_n - u) ds \right| |\varphi| dt \tag{2.21}
\]

\[
\leq \int_\mathbb{R} \left( \int_0^1 |V_{qq}(t, \chi + u + s(u_n - u))| |u_n - u| ds \right) |\varphi| dt.
\]

But (as before), we know that for all \( s \in [0, 1] \) and \( t \in \mathbb{R} \), we have

\[
|\chi(t) + u(t) + s(u_n(t) - u(t))| \leq \|\chi\|_{L^\infty} + 2\|u\|_{L^\infty} + \|u_n\|_{L^\infty} \leq \tilde{C}, \tag{2.22}
\]

where \( \tilde{C} \) depends on the \( L^\infty \) norm of \( \chi \), \( u \) and the \( L^\infty \) bounds on \( u_n \). This then implies that

\[
|V_{qq}(t, \chi + s + s(u_n - u))| \leq \max_{0 \leq t \leq 1, |x| \leq C} |V_{qq}(t, x)| =: M, \tag{2.23}
\]

so combining (2.21) and (2.23), we have

\[
\left| \int_\mathbb{R} \langle V_q(t, \chi + u_n) - V_q(t, \chi + u), \varphi \rangle dt \right|
\]

\[
\leq \int_\mathbb{R} \left( \int_0^1 \|V_{qq}(t, \chi + u + s(u_n - u))\| |u_n - u| ds \right) |\varphi| dt
\]

\[
\leq \int_\mathbb{R} M|u_n - u| |\varphi| dt \tag{2.24}
\]

\[
\leq M\|u_n - u\|_{L^2}
\]
since $\|\varphi\|_{W^{1,2}} \leq 1$. Altogether, (2.19) and (2.24) imply that

$$\left| (K(u_n) - K(u))\varphi \right| \leq (M + 1)\|u_n - u\|_{L^2}$$

(2.25)

for some $M$ independent of $\varphi$.

Because (2.25) is independent of $\varphi$, we have

$$\sup_{\|\varphi\|_{W^{1,2}} \leq 1} \left| (K(u_n) - K(u))\varphi \right| \leq (M + 1)\|u_n - u\|_{L^2},$$

and so $K(u_n)$ converges strongly to $K(u)$. \qed

We now prove a corollary:

**Corollary 2.1.7.** Suppose that $I(\chi) < \infty$, and $\{u_n\} \subset E$ and $u \in E$ are such that

(i) $\|u_n - u\|_{L^2} \to 0$ as $n \to \infty$

(ii) $J'_\chi(u_n) \to 0$ as $n \to \infty$

(iii) $u_n$ is bounded in $L^\infty$

(iv) $J'_\chi(u) = 0$

Then, $\|u_n - u\|_E \to 0$ as $n \to \infty$.

**Proof.** By Proposition 2.1.6, we know that

$$J'_\chi(u)\varphi = \langle u, \varphi \rangle + K(u)$$

(2.26)

for all $\varphi \in E$. But then by (iv)

$$|\langle u_n - u, \varphi \rangle| \leq |J'_{\chi}(u_n)\varphi| + |J'_{\chi}(u)| + |(K(u_n) - K(u))\varphi|$$

$$\leq \|J'_{\chi}(u_n)\|_{E'}\|\varphi\|_E + \|K(u_n) - K(u)\|_{E'}\|\varphi\|_E$$

(2.27)

$$\leq \|J'_{\chi}(u_n)\|_{E'} + \|K(u_n) - K(u)\|_{E'}$$
for all $\varphi \in E$ with $\|\varphi\|_E = 1$. By (ii), the first term in (2.27) goes to 0 as $n \to \infty$.

Since $u_n \to u$ in $L^2$ by (i) and $u_n$ is bounded in $L^\infty$ by (iii), (ii) of Proposition 2.1.6 implies that the second term in (2.27) tends to 0 as $n \to \infty$. Since

$$
\|u_n - u\|_E = \sup_{\|\varphi\|_E = 1} |\langle u_n - u, \varphi \rangle|,
$$

(2.27) implies the Corollary. □

**Remark 2.1.8.** Suppose now that we have a sequence $\{v_n\} \subset \hat{E}$ such that:

(i) $v_n(-\infty) = 0$ for all $n$, and $v_n(\infty) = \xi_j$ for all $n$ (i.e. all the $v_n$ have the same asymptotics) and

(ii) $I(v_n) \to b$ and $I'(v_n) \to 0$, i.e. $v_n$ is a (PS) sequence for $I$.

Then, because of the boundedness of $I(v_n)$, we can show that $v_n$ is bounded in $\hat{E}$, and therefore (along a subsequence) $v_n \to v$ in $\hat{E}$. If this $v$ has the same asymptotics as the $v_n$ and $v$ solves (HS), then we can take $\chi := v$. (In order to get the proper asymptotic behavior of $v$, it will most likely be necessary to renormalize the sequence $v_n$ in an appropriate fashion.) Notice that if we take $u_n := v_n - v$ and $u = 0$, we have $J_\chi(u_n) \to b$ and $J_\chi'(u_n) \to 0$. If we can show that $\|u_n - u\|_{L^2} = \|v_n - v\|_{L^2} \to 0$, then Corollary 2.1.7 implies that $u_n \to u$ in $E$. Thus, to determine whether or not a (PS) sequence converges, it suffices to look at the $L^2$ convergence of $v_n$ to $v$.

For the remainder of this section, we consider a sequence $v_n$ that satisfies the conditions in the preceding remark. (One should bear in mind that $\xi$ might possibly equal 0.) Without loss of generality, we may assume that $I(v_n) \leq b+1$ for all $n \in \mathbb{N}$. 
Definition 2.1.9. For any $r < \frac{\eta}{2}$ (where $\eta$ is from Remark 2.1.1), we let

$$\alpha(r) := \inf_{0 \leq t \leq 1, x \not\in \cup_i B_r(\xi_i)} -V(t, x)$$

Notice that by our assumptions about the potential $V$, $\alpha(r) > 0$ for $r > 0$. We have the following fundamental lemma:

Lemma 2.1.10. [19], [22] Suppose $a < b \in \mathbb{R}$ are such that $v(t) \not\in B_r(\xi_j)$ for all $t \in (a, b)$ and $j = 1, 2, \ldots k$. Then:

$$\int_a^b \left( \frac{1}{2} |\dot{v}|^2 - V(t, v(t)) \right) dt \geq \sqrt{2\alpha(r)} |v(b) - v(a)|$$

For any $r > 0$, we let $B_r(K(V))$ denote the union of $B_r(\xi_j)$ over all $\xi_j \in K(V)$.

Definition 2.1.11. For every $n \in \mathbb{N}$, we let

$$B_n := \{ t \in \mathbb{R} | v_n(t) \in B_{\eta/2}(K(V)) \}.$$ 

Notice that $B_n$ is open, so it is the union of an at most countable number of open intervals. (Let us note that at the endpoints of these intervals, we must have $v_n \in \partial B_{\eta/2}(K(V)).$) Suppose that $(a_n, b_n)$ is one such interval, and that $b_n - a_n \geq (b + 2)(\frac{1}{\alpha(\eta/4)})$. If there is no $t \in (a_n, b_n)$ such that $v_n(t) \in B_{\eta/4}(K(V))$, then we must have $-V(t, v_n(t)) \geq \alpha(\frac{\eta}{4})$, and so:

$$b + 1 \geq I(v_n) \geq \int_{a_n}^{b_n} -V(t, v_n(t)) dt \geq \alpha(\eta/4)(b_n - a_n) \geq (b + 2),$$

which is impossible. Thus, on any interval whose length is larger than $(\frac{b+2}{\alpha(\eta/4)})$, there is at least one $t$ at which $v_n \in B_{\eta/4}(K(V))$. Thus, on any such interval, $v_n$ will have to travel from $\partial B_{\eta/2}(K(V))$ at the endpoints to $B_{\eta/4}(K(V))$, and so (using Lemma 2.1.10), on any such interval,

$$\int_{a_n}^{b_n} \left( \frac{1}{2} |v_n|^2 - V(t, v_n(t)) \right) dt \geq \frac{\eta}{4} \sqrt{\frac{\eta}{4}}.$$
Therefore, for any \( n \), there can be at most a finite number of such long intervals. Call this number \( l(n) \). Moreover, the number \( l(n) \) of such intervals must be bounded independent of \( n \). Now, let us consider the maximal intervals \((-\infty, \bar{t}_{n_1}), (t_{n_2}, \bar{t}_{n_2}), \ldots, (t_{n_l(n)}, \infty)\) in \( B_n \) that have the additional property that their lengths go to \( \infty \) as \( n \to \infty \). Since the number of such intervals \( l(n) \) is bounded independent of \( n \) for any PS sequence, we can assume that \( l(n) \) is the same for all large enough \( n \).

**Lemma 2.1.12.** \( t_{n+i} - \bar{t}_{n_i} \) is bounded independently of \( n, i \).

**Proof.** Suppose not. Then, we have

\[
(\bar{t}_{n_i}, t_{n_{i+1}}) = (B_n \cap (\bar{t}_{n_i}, t_{n_{i+1}})) \cup \{ t \in (\bar{t}_{n_i}, t_{n_{i+1}}) \mid v_n(t) \not\in B_{\eta/2}(K(V)) \}
\]

By Lemma 2.1.10, the measure of the second term must be bounded, since \( I(v_n) \leq b + 1 \) for all \( n \). Thus,

\[
\left| (B_n \cap (\bar{t}_{n_i}, t_{n_{i+1}})) \right| \to \infty
\]  

as \( n \to \infty \), where \( |A| \) denotes the measure of a set \( A \). Now,

\[
(B_n \cap (\bar{t}_{n_i}, t_{n_{i+1}})) = \bigcup_{i=1}^{j(n)} (a_i^n, b_i^n),
\]

where the \((a_i^n, b_i^n)\) are maximal intervals in which \( v_n(t) \in B_{\eta/2}(K(V)) \). Notice that because of the maximality of \((a_i^n, b_i^n)\), \( v_n \in \partial B_{\eta/2}(K(V)) \) at the endpoints of these intervals. Now,

\[
b_i^n - a_i^n \text{ is bounded independent of } n
\]  

because \((-\infty, \bar{t}_{n_1}), (t_{n_2}, \bar{t}_{n_2}), \ldots, (t_{n_l}, \infty)\) are the only subintervals of \( B_n \) whose lengths are unbounded in \( n \). Thus, in order for (2.28) to be true, we must have
\( j(n) \to \infty \) as \( n \to \infty \). Using Lemma 2.1.10, we cannot have \( v_n(t) \notin B_{\eta/4}(K(V)) \) for all \( t \in (B_n \cap (\bar{t}_{n_i}, \bar{t}_{n_{i+1}})) \) because of the boundedness of \( I(v_n) \). Therefore, (using \( \sqcup \) to denote a disjoint union),

\[
(B_n \cap (\bar{t}_{n_i}, \bar{t}_{n_{i+1}})) = \bigcup_{i}^{j_G(n)} (\tilde{a}_i^n, \tilde{b}_i^n) \bigcup_{i}^{j_B(n)} (c_i^n, d_i^n) \tag{2.31}
\]

where \( j_G(n) \) is the number of intervals where \( v_n(t) \in B_{\eta/4}(K(V)) \) for some \( t \in (\tilde{a}_i^n, \tilde{b}_i^n) \), and \( j_B(n) \) is the number of intervals where \( v_n \) avoids \( B_{\eta/4}(K(V)) \). By (2.29) and (2.31), we must have

\[
j_G(n) + j_B(n) = j(n) \to \infty \tag{2.32}
\]
as \( n \to \infty \). Thus, by (2.32), at least one of \( j_G(n), j_B(n) \to \infty \) as \( n \to \infty \).

Now, on each interval \( (\tilde{a}_i^n, \tilde{b}_i^n) \), \( v_n \) goes from \( \partial B_{\eta/2}(K(V)) \) at the endpoints to \( B_{\eta/4}(K(V)) \) somewhere inside. Thus by Lemma 2.1.10, on each of these \( j_G(n) \) intervals, we get a contribution of at least \( \eta/4 \sqrt{2\alpha(\eta/4)} \) to \( I(v_n) \). Since \( I(v_n) \) is bounded, \( j_G(n) \leq C_1 \) for some \( C_1 \), and so by (2.30),

\[
\left| \bigcup_{i}^{j_G(n)} (\tilde{a}_i^n, \tilde{b}_i^n) \right| \leq C_2, \tag{2.33}
\]
for some \( C_2 \). Thus, by (2.32), \( j_B(n) \to \infty \).

By Lemma 2.1.10, the length of any interval \( (c_i^n, d_i^n) \) where \( v_n \) avoids \( B_{\eta/4}(K(V)) \) must be bounded. By (2.28) and (2.33), we have

\[
\left| \bigcup_{i}^{j_B(n)} (c_i^n, d_i^n) \right| \to \infty. \tag{2.34}
\]

But then Definition 2.1.9 and (2.34) imply that

\[
I(v_n) \geq \sum_{i}^{j_G(n)} \int_{c_i^n}^{d_i^n} \alpha \left( \frac{\eta}{4} \right) dt = \alpha \left( \frac{\eta}{4} \right) \sum_{i}^{j_B(n)} (d_i^n - c_i^n) \to \infty
\]
as \( n \to \infty \), which contradicts the fact that \( I(v_n) \) is bounded. \( \square \)
Proposition 2.1.13. Suppose that there are only two such intervals: \((-\infty, \bar{t}_{n_1})\) and \((\bar{t}_{n_2}, \infty)\). Then, there is sequence \(k_n \in \mathbb{Z}\) such that

\[(i) \quad I(\tau_{k_n} v_n) \to b, \text{ and } I'(\tau_{k_n} v_n) \to 0\]

\[(ii) \quad \text{There is a } v \in \hat{E} \text{ solving (HS) such that } \|\tau_{k_n} v_n - v\|_{W^{1,2}(\mathbb{R})} \to 0\]

Proof. Pick \(k_n \in \mathbb{Z}\) such that \(\bar{t}_{n_1} + k_n \in [0, 1)\). We have \(\tau_{k_n} v_n(\bar{t}_{n_1} + k_n) = v_n(\bar{t}_{n_1}) \in \partial B_{\eta/2}(K(V))\), and \(I(\tau_{k_n} v_n) = I(v_n)\) (because of the \(\mathbb{Z}\) action). Moreover, \(\|I'(\tau_{k_n} v_n)\| = \|I'(v_n)\|\). Thus, (i) is proven.

Because of the choice of \(k_n\), we have \(0 \leq \bar{t}_{n_1} + k_n\), hence \(\tau_{k_n} v_n(0) \in B_{\eta/2}(0)\) for all \(n\). Because \(b + 1 \geq I(\tau_{k_n} v_n^1)\), we also have \(\|\tau_{k_n} \dot{v}_n\|_{L^2}\) bounded. Thus \(\tau_{k_n} v_n\) is bounded in \(\hat{E}\), and so (along a subsequence) \(\tau_{k_n} v_n \to v\) for some \(v \in \hat{E}\). In particular, this means that \(\tau_{k_n} \dot{v}_n \to \dot{v}\) in \(L^2\). In addition, since \(I(v_n) \leq b + 1\) for all \(n\) and \(v_n(-\infty) = 0\) for all \(n\), we know that \(\tau_{k_n} v_n\) is bounded in \(L^\infty\). Together, these two facts imply that \(\tau_{k_n} v_n\) converges to \(v\) in \(L^\infty_{loc}\). By the weak lower semicontinuity of \(I\) (see [18] and [19]), \(I(v) \leq b\). Next, we show that \(v\) solves (HS). To do this, it suffices to show that

\[I'(v) \varphi = \int_\mathbb{R} \langle \dot{v}, \varphi \rangle - \langle V_q(t, v), \varphi \rangle \, dt = 0 \quad (2.35)\]

for all \(\varphi \in C^\infty_c(\mathbb{R})\). Fix \(\varphi \in C^\infty_c(\mathbb{R})\). Then, we know that

\[I'(\tau_{k_n} v_n) \varphi = \int_\mathbb{R} \langle \tau_{k_n} \dot{v}_n, \varphi \rangle - \langle V_q(t, v_n), \varphi \rangle \, dt \to 0 \quad (2.36)\]

as \(n \to \infty\). Since \(\tau_{k_n} v_n \to v\) in \(L^\infty_{loc}\) and \(\varphi\) has compact support, we know that

\[\int_\mathbb{R} \langle V_q(t, \tau_{k_n} v_n), \varphi \rangle \, dt \to \int_\mathbb{R} \langle V_q(t, v), \varphi \rangle \, dt \quad (2.37)\]
as \( n \to \infty \). Moreover, since \( \tau_{k_n} \dot{v}_n \to \dot{v} \) in \( L^2 \), we know that

\[
\int_{\mathbb{R}} \langle \tau_{k_n} \dot{v}_n, \dot{\varphi} \rangle \, dt \to \int_{\mathbb{R}} \langle \dot{v}, \dot{\varphi} \rangle \, dt \tag{2.38}
\]
as \( n \to \infty \). Combining (2.36) - (2.38), we see that

\[
0 = \lim_{n \to \infty} I'(\tau_{k_n} v_n) \varphi = \lim_{n \to \infty} \left( \int_{\mathbb{R}} \langle \tau_{k_n} \dot{v}_n, \dot{\varphi} \rangle - \langle V_q(t, \tau_{k_n} v_n), \varphi \rangle \, dt \right)
\]
\[
= \int_{\mathbb{R}} \langle \dot{v}, \dot{\varphi} \rangle - \langle V_q(t, v), \varphi \rangle \, dt \tag{2.39}
\]
\[
= I'(v) \varphi.
\]

Since (2.39) holds for all \( \varphi \in C_c^\infty(\mathbb{R}) \), \( v \) solves (HS).

To get (ii), we want to apply Corollary 2.1.7. Let \( \chi := v, \ u_n := \tau_{k_n} v_n - v \) and \( u := 0 \). Notice that \( J'_\chi(u_n) = I'(\tau_{k_n} v_n) \to 0 \) as \( n \to \infty \). Moreover, we know that \( v_n \) is bounded in \( L^\infty \), since \( I(v_n) \leq b + 1 \), hence \( \|u_n\|_{L^\infty} \leq \|v_n\|_{L^\infty} \). Moreover, since \( v \) solves (HS), \( 0 = I'(v) = J'_\chi(u) \). Thus, to apply Corollary 2.1.7, we need to show that

\[
\|u_n - u\|_{L^2} = \|\tau_{k_n} v_n - v\|_{L^2} \to 0 \tag{2.40}
\]
as \( n \to \infty \). Once we have (2.40), Corollary 2.1.7 implies that

\[
\|u_n - u\|_E = \|\tau_{k_n} v_n - v\|_E \to 0 \tag{2.41}
\]
as \( n \to \infty \), which implies (ii) of Proposition 2.1.13.

First, we show that \( v \not\equiv \) constant. Passing to a subsequence, we must have \( \tilde{t}_{n_1} + k_n \to \tilde{t} \in [0,1] \). We claim that \( \tau_{k_n} v_n(\tilde{t}_{n_1} + k_n) \to v(\tilde{t}) \). If so, then since

\[
\tau_{k_n} v_n(\tilde{t}_{n_1} + k_n) = v_n(\tilde{t}_{n_1}) \in \partial B_{\eta/2}(K(V)) \text{ for all } n, \text{ we will have } v(\tilde{t}) \in \partial B_{\eta/2}(K(V)).
\]

This implies \( v \not\equiv \) constant, since \( I(v) < \infty \), and the only constant functions for
which \( I(v) < \infty \) are when \( v \equiv \xi \) for some \( \xi \in K(V) \). To prove our claim, note that

\[
|v(\bar{t}) - \tau_{k_n} v_n(\bar{t}_n + k_n)| \leq |v(\bar{t}) - \tau_{k_n} v_n(\bar{t})| + |\tau_{k_n} v_n(\bar{t}) - \tau_{k_n} v_n(\bar{t}_n + k_n)|.
\]

The first term goes to 0 because of the \( L^\infty_{\text{loc}} \) convergence of \( \tau_{k_n} v_n \) to \( v \). The second term goes to 0 because \( \{\tau_{k_n} v_n\} \) is equicontinuous (by the \( L^2 \) bound on the derivatives).

Next, we claim that \( v(\bar{t}) \in B_{\eta/2}(0) \) for all \( t < 0 \). To see this, note that (by the choice of \( k_n \) and the assumption about the behavior of our sequence at \( -\infty \)) \( \tau_{k_n} v_n(t) \in B_{\eta/2}(0) \) for all \( t < 0 \). Thus, by pointwise convergence, we must have \( v(t) \in \overline{B_{\eta/2}(0)} \) for all \( t < 0 \). Therefore, (since \( I(v) < \infty \)) we must have \( v(-\infty) = 0 = \tau_{k_n} v_n(-\infty) \) for all \( n \).

Suppose now that \( v_n(\infty) = \xi \) for all \( n \) and some \( \xi \in K(V) \). We want to show that \( v(t) \in \overline{B_{\eta/2}(\xi)} \) for all sufficiently large \( t \), which will imply (as above) that \( v(\infty) = \xi \). We know from Lemma 2.1.12 that \( (t_{n_2} - \bar{t}_{n_1}) \) is bounded, so \( t_{n_2} - \bar{t}_{n_1} \to t^* \). Because \( \bar{t}_{n_1} + k_n \to \bar{t} \), we have \( t_{n_2} + k_n = t_{n_2} - \bar{t}_{n_1} + \bar{t}_{n_1} + k_n \to \bar{t} + t^* \).

Suppose now that \( t > \bar{t} + t^* + 1 \). Then, for sufficiently large \( n \), we have \( t_{n_2} + k_n < \bar{t} + t^* + 1 < t \), hence \( t_{n_2} < t - k_n \) for all large \( n \). But then \( v_n(t - k_n) \in B_{\eta/2}(\xi) \), so \( \tau_{k_n} v_n(t) \in B_{\eta/2}(\xi) \) for all large \( n \), and so \( v(t) \in \overline{B_{\eta/2}(\xi)} \). Thus, \( v(\infty) = \xi \). Notice that the preceding two paragraphs prove the following

**Proposition 2.1.14.** There is an \( N \) such that if \( n > N \), then

\[
\begin{align*}
(i) & \quad \tau_{k_n} v_n(t) \in B_{\eta/2}(0) \text{ for } t < 0 \\
(ii) & \quad \tau_{k_n} v_n(t) \in B_{\eta/2}(\xi) \text{ for } t > \bar{t} + t^* + 1
\end{align*}
\]
So, \( \tau_{k_n} v_n \) and \( v \) have the same asymptotics. To prove part (ii), by Corollary 2.1.7, it suffices to show that \( \| \tau_{k_n} v_n - v \|_{L^2} \to 0 \). If this is not the case, there must be a \( \delta > 0 \) such that on a subsequence (which we relabel)

\[
\| \tau_{k_n} v_n - v \|_{L^2} \geq \delta. \tag{2.42}
\]

For any \( \rho < \frac{n}{4} \), let

\[
\underline{t}(\rho) := \sup \{ t \mid v(s) \in B_\rho(0) \text{ for } s < t \}
\]

\[
\bar{t}(\rho) := \inf \{ t \mid v(s) \in B_\rho(\xi) \text{ for } s > t \}. \tag{2.43}
\]

Notice that we have \( \underline{t}(\rho) \to -\infty \) and \( \bar{t}(\rho) \to \infty \) as \( \rho \to 0 \). Next, let \( A(\rho) := (\underline{t}(\rho), \bar{t}(\rho)) \), and \( B(\rho) := \mathbb{R} - A(\rho) \). For a fixed \( \rho > 0 \), the local uniform convergence of \( \tau_{k_n} v_n \) to \( v \) implies that

\[
\| \tau_{k_n} v_n - v \|_{L^2(A(\rho))} \to 0 \tag{2.44}
\]

as \( n \to \infty \). We will now show that there is a \( \rho > 0 \) such that for all \( n > N(\rho) \), one has

\[
\| \tau_{k_n} v_n - v \|_{L^2(-\infty, \underline{t}(\rho))} < \frac{\delta}{4} \tag{2.45}
\]

and

\[
\| \tau_{k_n} v_n - v \|_{L^2(\bar{t}(\rho), \infty)} < \frac{\delta}{4}. \tag{2.46}
\]

Now by Lemma 2.1.4, we may pick \( \rho \) suitably small that \( \| v \|_{L^2(-\infty, \underline{t}(\rho))} < \delta/8 \) and

\[
\| v - \xi \|_{L^2(\bar{t}(\rho), \infty)} < \delta/8. \]

In order to show (2.45) and (2.46), it suffices to show that
there is an $N(\rho)$ such that for $n > N(\rho)$, one has
\[ \| \tau_{k_n} v_n \|_{W^{1.2}(-\infty, \ell(\rho))} < \frac{\delta}{8} \] (2.47)

and
\[ \| \tau_{k_n} v_n - \xi \|_{W^{1.2}(\bar{\ell}(\rho), \infty)} < \frac{\delta}{8} \] (2.48)

To do this, we need the following proposition

**Proposition 2.1.15.** For any fixed $\rho$, there is an $N(\rho)$ such that for $n > N(\rho)$

(i) $\tau_{k_n} v_n(t) \in B_{2\rho}(0)$ for $t < \ell(\rho)$

(ii) $\tau_{k_n} v_n(t) \in B_{2\rho}(\xi)$ for $t > \bar{\ell}(\rho)$

Assuming Proposition 2.1.15, let us continue with the proof of Proposition 2.1.13. We have

\[
\frac{1}{2} \| \tau_{k_n} \dot{v}_n \|_{L^2(-\infty, \ell(\rho))}^2 = \int_{-\infty}^{\ell(\rho)} \frac{1}{2} |\tau_{k_n} \dot{v}_n|^2 dt
\]

\[
= \frac{1}{2} \int_{-\infty}^{\ell(\rho)} \left( |\tau_{k_n} v_n|^2 - \langle V_q(t, \tau_{k_n} v_n), \tau_{k_n} v_n \rangle \right) dt
\]

\[
+ \int_{-\infty}^{\ell(\rho)} \left( \frac{1}{2} \langle V_q(t, \tau_{k_n} v_n), \tau_{k_n} v_n \rangle - V(t, \tau_{k_n} v_n) \right) dt + \int_{-\infty}^{\ell(\rho)} V(t, \tau_{k_n} v_n) dt
\] (2.49)

and so

\[
\frac{1}{2} \| \tau_{k_n} \dot{v}_n \|_{L^2(-\infty, \ell(\rho))}^2 + \int_{-\infty}^{\ell(\rho)} -V(t, \tau_{k_n} v_n) dt
\]

\[
= \frac{1}{2} \int_{-\infty}^{\ell(\rho)} \left( |\tau_{k_n} \dot{v}_n|^2 - \langle V_q(t, \tau_{k_n} v_n), \tau_{k_n} v_n \rangle \right) dt
\]

\[
+ \int_{-\infty}^{\ell(\rho)} \left( \frac{1}{2} \langle V_q(t, \tau_{k_n} v_n), \tau_{k_n} v_n \rangle - V(t, \tau_{k_n} v_n) \right) dt
\] (2.50)
By Proposition 2.1.14, Remark 2.1.1 and the fact \( \tau_{k_n} v_n(-\infty) = 0 \), we have
\[
\beta_1 \int_{-\infty}^{t(\rho)} |\tau_{k_n} v_n|^2 < \int_{-\infty}^{t(\rho)} -V(t, \tau_{k_n} v_n) dt.
\] (2.51)

Hence, by (2.50) and (2.51), we have
\[
\frac{1}{2} \| \tau_{k_n} \dot{v}_n \|_{L^2(-\infty, t(\rho))} + \beta_1 \int_{-\infty}^{t(\rho)} |\tau_{k_n} v_n|^2 dt
\leq \frac{1}{2} \int_{-\infty}^{t(\rho)} (|\tau_{k_n} \dot{v}_n|^2 - \langle V_q(t, \tau_{k_n} v_n), \tau_{k_n} v_n \rangle) dt
\] (2.52)
\[
+ \int_{-\infty}^{t(\rho)} \left( \frac{1}{2} \langle V_q(t, \tau_{k_n} v_n), \tau_{k_n} v_n \rangle - V(t, \tau_{k_n} v_n) \right) dt.
\]

Now, by the assumptions about our potential \( V \), if \( |x| \leq 2\rho \), then
\[
\left| \frac{1}{2} \langle V_q(t, x), x \rangle - V(t, x) \right| \leq \epsilon(\rho) |x|^2,
\]
where \( \epsilon(\rho) \to 0 \) as \( \rho \to 0 \). By Proposition 2.1.15, we have for all \( n > N(\rho) \) that
\( \tau_{k_n} v_n(t) \in B_{2\rho}(0) \) for \( t < t(\rho) \), and so
\[
\left| \frac{1}{2} \langle V_q(t, \tau_{k_n} v_n(t)), \tau_{k_n} v_n(t) \rangle - V(t, \tau_{k_n} v_n(t)) \right| \leq \epsilon(\rho) |\tau_{k_n} v_n(t)|^2
\] (2.53)
for all \( n > N(\rho) \) and \( t < t(\rho) \). Thus, if we take \( \rho \) sufficiently small that \( \epsilon(\rho) < \frac{1}{2} \beta_1 \), then using (2.53), we can move the last term on the right in (2.52) to the left and absorb to get
\[
\frac{1}{2} \min(\beta_1, 1) \| \tau_{k_n} v_n \|_{W^{1,2}(-\infty, t(\rho))}^2 \leq \frac{1}{2} \int_{-\infty}^{t(\rho)} (|\tau_{k_n} \dot{v}_n|^2 - \langle V_q(t, \tau_{k_n} v_n), \tau_{k_n} v_n \rangle) dt
\] (2.54)
for all \( n > N(\rho) \). Now, we want to show that the right hand side is \( < \frac{\delta}{8} \). Let
\[
\psi_n(t) := \begin{cases} 
\tau_{k_n} v_n(t) & \text{for } t < t(\rho) \\
(\rho(t) + 1) - t \tau_{k_n} v_n(t(\rho)) & \text{for } t(\rho) \leq t \leq \rho(t) + 1 \\
0 & \text{for } t > \rho(t) + 1
\end{cases}
\]
We claim now that \( \psi_n \) is bounded in \( W^{1,2} \) independently of \( n \) and \( \rho \). To see this, note that by (2.43) and (2.51)

\[
\| \psi_n \|_{W^{1,2}}^2 = \int_{-\infty}^{t(\rho)} |\tau_{kn} \dot{v}_n|^2 + |\tau_{kn} v_n|^2 dt \\
+ \int_{t(\rho)}^{t(\rho)+1} \left( |\tau_{kn} v_n(t(\rho))|^2 + \left( (t(\rho) + 1) - t \right)^2 |\tau_{kn} v_n(t(\rho))|^2 \right) dt \\
\leq \int_{-\infty}^{t(\rho)} \left( |\tau_{kn} \dot{v}_n|^2 - \frac{1}{\beta_1} V(t, \tau_{kn} v_n) \right) dt + 2|\tau_{kn} v_n(t(\rho))|^2 \\
\leq \left( 2 + \frac{1}{\beta_1} \right) I(\tau_{kn} v_n) + \eta^2 < \left( 2 + \frac{1}{\beta_1} \right) (b + 1) + \eta^2
\]

since \( \rho < \eta/4 \). Notice that this implies that \( I'(\tau_{kn} v_n) \psi_n \to 0 \) as \( n \to \infty \), since

\[
|I'(\tau_{kn} v_n)\psi_n| \leq \|I'(\tau_{kn} v_n)\|\|\psi_n\|_{W^{1,2}} \leq C\|I'(\tau_{kn} v_n)\|.
\]

But

\[
I'(\tau_{kn} v_n) \psi_n = \int_{-\infty}^{t(\rho)} \left( |\tau_{kn} \dot{v}_n|^2 - \langle V(t, \tau_{kn} v_n), \tau_{kn} v_n \rangle \right) dt \\
+ \int_{t(\rho)}^{t(\rho)+1} \langle \tau_{kn} \dot{v}_n(t), \tau_{kn} v_n(t(\rho)) \rangle dt \\
+ \int_{t(\rho)}^{t(\rho)+1} -\langle V(t, \tau_{kn} v_n), \left( (t(\rho) + 1) - t \right) \tau_{kn} v_n(t(\rho)) \rangle dt
\]

Let us turn our attention to the last two terms in (2.57). By (2.43), we have

\[
\left| \int_{t(\rho)}^{t(\rho)+1} \langle \tau_{kn} \dot{v}_n(t), \tau_{kn} v_n(t(\rho)) \rangle dt \right| \leq \left( \int_{t(\rho)}^{t(\rho)+1} |\tau_{kn} \dot{v}_n| |\tau_{kn} v_n(t(\rho))| dt \right) \\
\leq 2\rho \left( \int_{\mathbb{R}} |\tau_{kn} \dot{v}_n|^2 dt \right)^{\frac{1}{2}} \leq C\rho
\]
and

\[ \left| \int_{\mathcal{L}(\rho)}^{\mathcal{L}(\rho)+1} \left( \langle V_q(t, \tau_n v_n), ((\mathcal{L}(\rho) + 1) - t) \tau_n v_n(t(\rho)) \rangle \right) dt \right| \leq \max_{0 \leq t \leq 1, |x| \leq \eta} |V_q(t, x)| |\tau_n v_n(t(\rho))| \leq K \rho \quad \text{(because of Prop. 2.1.15)} \tag{2.59} \]

Thus, combining (2.58) and (2.59), the second term of (2.57) is bounded by $\tilde{C} \rho$ for some $\tilde{C}$ independent of $\rho$. Because of (2.54) and (2.57), we have shown the following:

\[ \min(1, \beta_1) \|\tau_n v_n\|_{W^{1,2}(-\infty, \mathcal{L}(\rho))}^2 \leq \int_{-\infty}^{\mathcal{L}(\rho)} \left( \|\tau_n v_n\|^2 - \langle V_q(t, \tau_n v_n), \tau_n v_n \rangle \right) dt \]
\[ = I'(\tau_n v_n) \psi_n - \int_{\mathcal{L}(\rho)}^{\mathcal{L}(\rho)+1} \langle \tau_n v_n(t), \tau_n v_n(t(\rho)) \rangle dt \]
\[ - \int_{\mathcal{L}(\rho)}^{\mathcal{L}(\rho)+1} \langle V_q(t, \tau_n v_n), ((\mathcal{L}(\rho) + 1) - t) \tau_n v_n(t(\rho)) \rangle dt \tag{2.60} \]

By (2.56), $I'(\tau_n) \psi_n \to 0$ as $n \to \infty$. Moreover, the last two terms in (2.60) are bounded by $\tilde{C} \rho$. Thus, taking $\rho$ suitably small, we have for all sufficiently large $n$ that $\|\tau_n v_n\|_{W^{1,2}(-\infty, \mathcal{L}(\rho))} < \frac{\delta}{8}$.

To finish the proof of Proposition 2.1.13, it suffices then to show that there is a $\rho$ such that for all $n > N(\rho)$ we have $\|\tau_n v_n - \xi\|_{W^{1,2}(\mathcal{L}(\rho), \infty)} \leq \frac{\delta}{8}$. This is proved in an analogous fashion to the previous case, with a some added complication. By Lemma 2.1.4 we have $\tau_n v_n - \xi \in L^2(0, \infty)$ because $\tau_n v_n(\infty) = \xi$ and $I(\tau_n v_n)$ is finite. By Proposition 2.1.14 we have for all sufficiently large $t$ and $n$ that $\tau_n v_n(t) \in B_{\eta/2}(\xi)$, and so by Remark 2.1.1, we have

\[ \beta_1 |\tau_n v_n(t) - \xi|^2 \leq -V(t, \tau_n v_n(t)). \tag{2.61} \]
As before we have
\[
\frac{1}{2} \| \tau_{kn} \dot{v}_n \|_{L^2(\bar{t}(\rho), \infty)}^2 = \frac{1}{2} \int_{\bar{t}(\rho)}^{\infty} \left( |\tau_{kn} \dot{v}_n|^2 - \langle V_q(t, \tau_{kn} v_n), \tau_{kn} v_n - \xi \rangle \right) dt \\
+ \int_{\bar{t}(\rho)}^{\infty} \left( \frac{1}{2} \langle V_q(t, \tau_{kn} v_n), \tau_{kn} v_n - \xi \rangle - V(t, \tau_{kn} v_n) \right) dt \tag{2.62}
\]
\[
+ \int_{\bar{t}(\rho)}^{\infty} V(t, \tau_{kn} v_n) dt
\]

But, by (2.61), we know that
\[
\int_{\bar{t}(\rho)}^{\infty} V(t, \tau_{kn} v_n) dt \leq -\beta_1 \int_{\bar{t}(\rho)}^{\infty} |\tau_{kn} v_n - \xi|^2 dt. \tag{2.63}
\]

Hence, we have
\[
\frac{1}{2} \| \tau_{kn} \dot{v}_n \|_{L^2(\bar{t}(\rho), \infty)}^2 + \beta_1 \| \tau_{kn} v_n - \xi \|_{L^2(\bar{t}(\rho), \infty)}^2 \leq \frac{1}{2} \int_{\bar{t}(\rho)}^{\infty} |\tau_{kn} \dot{v}_n|^2 - \langle V_q(t, \tau_{kn} v_n), \tau_{kn} v_n - \xi \rangle dt \tag{2.64}
\]
\[
+ \int_{\bar{t}(\rho)}^{\infty} \left( \frac{1}{2} \langle V(t, \tau_{kn} v_n), \tau_{kn} v_n - \xi \rangle - V(t, \tau_{kn} v_n) \right) dt
\]

Now, if $|x - \xi| \leq 2\rho$, then
\[
\left| \frac{1}{2} \langle V_q(t, x), x - \xi \rangle - V(t, x) \right| \leq \epsilon(\rho) |x - \xi|^2, \tag{2.65}
\]
where $\epsilon(\rho) \to 0$ as $\rho \to 0$. Therefore, the last term on the right in (2.64) is then bounded by $\epsilon(\rho) \|\tau_{kn} v_n - \xi\|_{L^2(\bar{t}(\rho), \infty)}$. Taking $\rho$ sufficiently small, we have (as before)
\[
\frac{1}{2} \min(1, \beta_1) \| \tau_{kn} v_n - \xi \|_{W^{1,2}(\bar{t}(\rho), \infty)}^2 \leq \frac{1}{2} \int_{\bar{t}(\rho)}^{\infty} |\tau_{kn} \dot{v}_n|^2 - \langle V_q(t, \tau_{kn} v_n), \tau_{kn} v_n - \xi \rangle dt. \tag{2.66}
\]

We proceed as before: Let
\[
\psi_n(t) := \begin{cases} 
\tau_{kn} v_n(t) - \xi & \text{for } t > \bar{t}(\rho) \\
(t - (\bar{t}(\rho) - 1))(\tau_{kn} v_n(\bar{t}(\rho)) - \xi) & \text{for } \bar{t}(\rho) - 1 \leq t \leq \bar{t}(\rho) \\
0 & \text{for } t < \bar{t}(\rho) - 1
\end{cases}
\]
We have that $\|\psi_n\|_{W^{1,2}}$ is bounded independent of $n$ and $\rho$. The proof of this fact is the same as before, bearing in mind that

$$|\tau_{k_n} v_n(t) - \xi|^2 \leq -\frac{1}{\beta_1} V(t, \tau_{k_n} v_n(t))$$

for $t > \bar{t}(\rho)$. Thus, $I' (\tau_{k_n} v_n) \psi_n \to 0$ as $n \to \infty$. We have

$$I' (\tau_{k_n} v_n) \psi_n = \int_{\bar{t}(\rho)}^{\infty} (|\tau_{k_n} \dot{v}_n|^2 - \langle V_q(t, \tau_{k_n} v_n), \tau_{k_n} v_n - \xi \rangle) \, dt$$

$$+ \int_{\bar{t}(\rho)-1}^{\bar{t}(\rho)} \langle \tau_{k_n} \dot{v}_n, \tau_{k_n} v_n(\bar{t}(\rho)) - \xi \rangle \, dt$$

$$- \int_{\bar{t}(\rho)-1}^{\bar{t}(\rho)} \langle V_q(t, \tau_{k_n} v_n), (t - \bar{t}(\rho) + 1)(\tau_{k_n} v_n(\bar{t}(\rho)) - \xi) \rangle \, dt$$

We turn our attention to the second term. Using Proposition 2.1.15 and (2.43), we have

$$\left| \int_{\bar{t}(\rho)-1}^{\bar{t}(\rho)} \langle \tau_{k_n} \dot{v}_n, \tau_{k_n} v_n(\bar{t}(\rho)) - \xi \rangle \, dt \right| \leq \left| \tau_{k_n} v_n(\bar{t}(\rho)) - \xi \right| \left( \int_{\bar{t}(\rho)-1}^{\bar{t}(\rho)} |\tau_{k_n} \dot{v}_n|^2 \, dt \right)^{\frac{1}{2}}$$

$$\leq (2\rho) \| \tau_{k_n} \dot{v}_n \|_{L^2}$$

$$\leq (2\rho) (2I(\tau_{k_n} v_n))$$

$$\leq C\rho \text{ for } C \text{ independent of } \rho \text{ and } n$$

Similarly

$$\left| \int_{\bar{t}(\rho)-1}^{\bar{t}(\rho)} \langle V_q(t, \tau_{k_n} v_n), (t - \bar{t}(\rho) + 1)(\tau_{k_n} v_n(\bar{t}(\rho)) - \xi) \rangle \right|$$

$$\leq \max_{0 \leq t \leq 1, |x| \leq \eta} \left| V_q(t, x) \right| \left| \tau_{k_n} v_n(\bar{t}(\rho)) - \xi \right|$$

$$\leq (2\rho) \max_{0 \leq t \leq 1, |x| \leq \eta} \left| V_q(t, x) \right|$$

$$\leq \tilde{C}\rho \text{ (for } \tilde{C} \text{ independent of } n \text{ and } \rho)$$
The rest of the argument runs as before.

Thus, we must have \( \|\tau_{k_n}v_n - v\|_{L^2} \to 0 \), and so by Corollary 2.1.7, we have (ii) of the lemma.

Next, we have to prove Proposition 2.1.15. To do this, we will need some \( L^\infty \) estimates on a Palais-Smale sequence.

**Lemma 2.1.16.** Suppose that \( v_n \) is a sequence in \( \hat{E} \) such that \( I(v_n) \to b > 0 \) and \( I'(v_n) \to 0 \). Moreover, suppose that \( v_n(-\infty) = \xi_j \) for all \( n \), and \( v_n(\infty) = \xi_i \) for all \( n \). Then for all sufficiently large \( n \), \( \|v_n - \xi_j\|_{L^\infty} \geq \eta/2 \).

**Proof.** Notice that the result is true if \( \xi_i \neq \xi_j \), since \( |\xi_i - \xi_j| \geq \eta \). Thus, suppose that \( \xi_i = \xi_j \), and that \( \|v_n - \xi_j\|_{L^\infty} < \eta/2 \) for all large \( n \). We will now derive a contradiction. We have

\[
b + 1 \geq I(v_n) = \int_{\mathbb{R}} \left( \frac{1}{2} |\dot{v}_n|^2 - V(t, v_n) \right) dt
\]

\[
\geq \int_{\mathbb{R}} \left( \frac{1}{2} |\dot{v}_n|^2 + \beta_1 |v_n - \xi_j|^2 \right) dt
\]

\[
\geq \min \left\{ \frac{1}{2}, \beta_1 \right\} \int_{\mathbb{R}} \left( |\dot{v}_n|^2 + |v_n - \xi_j|^2 \right) dt
\]

Thus,

\[ \|v_n - \xi_j\|_{W^{1,2}} \leq \frac{b + 1}{\min \{ \frac{1}{2}, \beta_1 \}} \]

But then \( I'(v_n)(v_n - \xi_j) \to 0 \) as \( n \to 0 \). But we have

\[
I'(v_n)(v_n - \xi_j) = \int_{\mathbb{R}} \left( |\dot{v}_n|^2 - \langle V_q(t, v_n^1), v_n^1 - \xi_j \rangle \right) dt
\]

\[
\geq \int_{\mathbb{R}} \left( |\dot{v}_n|^2 + \beta_1 |v_n - \xi_j|^2 \right) dt.
\]

\[
(2.70)
\]
In (2.71), we use the fact that \( \|v_n - \xi_j\|_{L^\infty} < \eta/2 \). Therefore, we have

\[
I'(v_n)(v_n - \xi_j) \geq \min(1, \beta_1)\|v_n - \xi_j\|^2_{W^{1,2}} \tag{2.72}
\]

Thus, we must have \( \|v_n - \xi_j\|_{W^{1,2}} \to 0 \) as \( n \to 0 \). But this means (recalling the definition of \( J_\chi(u) \) for \( u \in W^{1,2} \)) we have

\[ I(v_n) = I(v_n - \xi_j + \xi_j) = J_{\xi_j}(v_n - \xi_j) \to J_{\xi_j}(0) = I(\xi_j) = 0, \]

a contradiction. \( \square \)

**Corollary 2.1.17.** If \( v \) is a non-constant solution to (HS) such that \( I(v) < \infty \), then

\[ \|v - v(\infty)\|_{L^\infty(\mathbb{R})} \geq \eta/2 \]

and

\[ \|v - v(-\infty)\|_{L^\infty(\mathbb{R})} \geq \eta/2 \]

**Proof.** If the statement is false, then there is a nonconstant solution \( v \) of (HS) such that \( I(v) < \infty \) and \( \|v - v(\infty)\|_{L^\infty} < \eta/2 \). By Lemma 2.1.3, we know that \( v(\infty) \in K(V) \). Without loss of generality, we may assume that \( v(\infty) = 0 \). The sequence \( u_n = v \) is a Palais-Smale sequence for \( I \), and \( I(u_n) = I(v) \) for all \( n \). We claim now that \( I(v) > 0 \), which would then contradict Lemma 2.1.16. Pick \( t_1 < t_2 \) such that \( |v(t_1)| = \|v\|_{L^\infty} > 0 \), and for all \( t \in (t_1, t_2) \), \( v(t) \in B_{\|v\|_{L^\infty}/2}(0) \). Then by Lemma 2.1.10 we have

\[ I(v) \geq \sqrt{2\alpha \left( \frac{\|v\|_{L^\infty}}{2} \right) |v(t_2) - v(t_1)|} > 0, \]

which finishes the proof. \( \square \)
Corollary 2.1.18. There is a $\nu > 0$ such that if $I'(v) = 0$ and $v$ is non-constant, then $I(v) > \nu$.

**Proof.** If there is no such $\nu$, then there is a sequence of non-constant critical points $v_n$ such that $I(v_n) \to 0$. By picking a subsequence, we may as well assume that $v_n(-\infty) = 0$ and $v_n(\infty) = \xi_i$ for all $n$. If $v_n(-\infty) \neq v_n(\infty)$, then the $v_n$ must travel between two equilibria, so by Lemma 2.1.10, $I(v_n) \geq \frac{\eta}{2} \sqrt{2\alpha \left( \frac{\eta}{2} \right)}$, which contradicts $I(v_n) \to 0$. Thus, we must have $v_n(\infty) = v_n(-\infty) = 0$ for all $n$. But then Lemma 2.1.16 implies that $\|v_n\|_{L^\infty} \geq \frac{\eta}{2}$, so $v_n$ must travel from $\partial B_{\eta/4}(0)$ to $\partial B_{\eta/2}(0)$. Thus, by Lemma 2.1.10, we must have $I(v_n) \geq \frac{\eta}{4} \sqrt{2\alpha \left( \frac{\eta}{4} \right)}$, which is also impossible. \[\square\]

Now, we are ready to prove Proposition 2.1.15. Recall what we are trying to prove: For any fixed $\rho$, there is an $N(\rho)$ such that for $n > N(\rho)$

(i) $\tau_{k_n}v_n(t) \in B_{2\rho}(0)$ for $t < \bar{t}(\rho)$

(ii) $\tau_{k_n}v_n(t) \in B_{2\rho}(\xi)$ for $t > \bar{t}(\rho)$,

where $v_n$ is a sequence with $v_n(-\infty) = 0$ and $v_n(\infty) = \xi$ for all $n$, and $I(v_n) \to b > 0$, $I'(v_n) \to 0$. ($t(\rho), \bar{t}(\rho)$ are defined in (2.43).) For simplicity, we write $v_n^1$ for $\tau_{k_n}v_n$. Thus, we have $v_n^1(t) = v_n(t - k_n)$.

**Proof.** Notice that by Proposition 2.1.14 we have already shown that if $\rho$ is small enough that $\bar{t}(\rho) > \bar{t} + t^* + 1$, then for all big enough $n$, we have

(i') $v_n^1(t) \in B_{\eta/2}(0)$ for $t < \bar{t}(\rho)$

(ii') $v_n^1(t) \in B_{\eta/2}(\xi)$ for $t > \bar{t}(\rho)$
Assume that (i) is false. We now want to get a contradiction. Since (i) is false, there must be a subsequence \( n_j \to \infty \) such that for every \( j \) there is a \( t_j < \tilde{t}(\rho) \) such that \( \tau_{k_n} v_{n_j}(t_j) \not\in B_{2\rho}(0) \). We claim that \( t_j \to -\infty \) as \( j \to \infty \). If not, then we must have \( t_j \to \hat{t} \leq \tilde{t}(\rho) \) on a subsequence. Now, we claim that \( v(\hat{t}) = \lim_{j \to \infty} v_{n_j}(t_j) \).

To see this:

\[
|v(\hat{t}) - v_{n_j}(t_j)| \leq |v(\hat{t}) - v_{n_j}(\hat{t})| + |v_{n_j}(t) - v_{n_j}(t_j)|
\]

(2.73)

The first term on the right in (2.73) \( \to 0 \) as \( j \to \infty \) because of pointwise convergence (indeed, we have \( L^\infty_{\text{loc}} \) convergence of \( v_{n_j} \) to \( v \)). The second term on the right in (2.73) also \( \to 0 \) because \( \{\tau_{k_n} v_n\} \) is equicontinuous. But this implies that \( |v(\hat{t})| \geq 2\rho \), which contradicts the definition of \( \tilde{t}(\rho) \) from (2.43). Now, let

\[
q_j(t) := \begin{cases} 
  v_{n_j}^1(t) & \text{for } t < \tilde{t}(\rho) \\
  (\tilde{t}(\rho) + 1 - t) (v_{n_j}^1(\tilde{t}(\rho))) & \text{for } \tilde{t}(\rho) \leq t \leq \tilde{t}(\rho) + 1 \\
  0 & \text{for } t > \tilde{t}(\rho)
\end{cases}
\]

We have \( \|q_j\|_{L^\infty} < \eta/2 \), since for \( t < \tilde{t}(\rho) \), \( q_j(t) = v_{n_j}^1(t) \in B_{\eta/2}(0) \), while for \( t \) between \( \tilde{t}(\rho) \) and \( \tilde{t}(\rho) + 1 \), \( q_j(t) \) is a convex combination of elements of \( B_{\eta/2}(0) \). Notice that we have \( q_j \in W^{1,2} \), and \( \|q_j\|_{W^{1,2}} \) is bounded independently of \( j \) and \( \rho \). (The proof of this is similar to showing the boundedness of \( \psi_n \) in Proposition 2.1.13, where we use Proposition 2.1.14.) Pick \( k_j \in \mathbb{Z} \) such that \( t_j + k_j \in [0, 1) \), and consider the sequence \( q_j^1(t) := q(t - k_j) \). Notice that this means that \( q_j^1(t_j + k_j) = q_j(t_j) = v_{n_j}^1(t_j) \not\in B_{2\rho}(0) \). But we also have \( \|q_j\|_{W^{1,2}} = \|q_j^1\|_{W^{1,2}} \), so (along a subsequence) \( q_j^1 \to q \) in \( W^{1,2} \). Now, we also have \( t_j + k_j \to \hat{t} \in [0, 1] \), and \( q_j^1 \to q \) in \( L^\infty_{\text{loc}} \), so we have

\[
|q(\hat{t}) - q_j^1(t_j + k_j)| \leq |q(\hat{t}) - q_j^1(\hat{t})| + |q_j^1(t) - q_j^1(t_j + k_j)|
\]

(2.74)
As in (2.73), the right side in (2.74) → 0 as \( j \to \infty \), hence \( |q(\hat{t})| \geq 2 \rho \), so \( q \) is not a constant. Because of the \( L^\infty \) bounds on \( q_j \), we must also have \( \|q\|_{L^\infty} \leq \eta/2 \). Moreover, by the weak lower semi-continuity of \( I \), we have \( I(q) < \infty \). Thus, if we can show that \( q \) satisfies (HS), we’ll have a contradiction to Corollary 2.1.17. It suffices to show

\[
\int_{\mathbb{R}} (\langle \dot{q}, \varphi \rangle - \langle V_q(t, q), \varphi \rangle) \, dt = 0
\]

for all \( \varphi \in C_c^\infty(\mathbb{R}) \). So, suppose that \( \varphi \in C_c^\infty(\mathbb{R}) \), and that \( \text{supp}(\varphi) \subset (a, c) \), and pick \( j \) so large that \( c - k_j < \hat{t}(\rho) \). (That this is possible relies on the fact that \( k_j \to \infty \), which follows from the fact that \( t_j \to -\infty \).) Then, changing variables, using the 1-periodicity of \( V \) and noting that because \( c - k_j < \hat{t}(\rho) \), we have

\[
\int_{\mathbb{R}} \langle \dot{q}, \varphi \rangle - \langle V_q(t, q), \varphi \rangle \, dt = \int_{\mathbb{R}} \langle \dot{q}_j(t), \varphi(t) \rangle - \langle V_q(t, q), \varphi(t) \rangle \, dt
\]

\[
= \int_a^c \langle \dot{q}_j(t), \varphi(t) \rangle - \langle V_q(t, q), \varphi(t) \rangle \, dt = \int_{a-k_j}^{c-k_j} \langle \dot{q}_j(t), \varphi(t+k_j) \rangle - \langle V_q(t, q), \varphi(t+k_j) \rangle \, dt
\]

\[
= \int_a^c \langle \dot{q}_j(t), \varphi(t+k_j) \rangle - \langle V_q(t, q), \varphi(t+k_j) \rangle \, dt
\]

\[
= \int_{\mathbb{R}} \langle \dot{v}_{n_j}^1, \varphi_j(t) \rangle - \langle V_q(t, v_{n_j}^1), \varphi_j(t) \rangle \, dt
\]

\[
= I'(v_{n_j}^1) \varphi_j,
\]

where \( \varphi_j(t) := \varphi(t+k_j) \). Now \( \|\varphi_j\|_{W^{1,2}} = \|\varphi\|_{W^{1,2}} \), so we have \( I'(v_{n_j}^1) \varphi_j \to 0 \) as \( j \to \infty \). Now, by the weak convergence of \( q_j^1 \) to \( q \), we have

\[
\int_{\mathbb{R}} \langle \dot{q}_j^1, \varphi \rangle \, dt \to \int_{\mathbb{R}} \langle \dot{q}, \varphi \rangle \, dt
\]
as \( j \to \infty \), and because of the \( L^\infty_{loc} \) convergence we have

\[
\int_\mathbb{R} \langle \dot{v}_j(t, q^1_j), \varphi \rangle dt \to \int_\mathbb{R} \langle \dot{v}(t, q), \varphi \rangle dt \tag{2.77}
\]
as \( j \to \infty \) because \( \varphi \) has compact support. Combining (2.76) and (2.77), we have

\[
\int_\mathbb{R} \langle \dot{q}, \dot{\varphi} \rangle - \langle \dot{V}(t, q), \varphi \rangle dt = \lim_{j \to \infty} I'(v^1_j) \varphi = 0
\]

This then contradicts Corollary 2.1.17, which proves (i). The proof of (ii) is similar.

\[ \square \]

Now, suppose we have a sequence \( v_n \) with \( I(v_n) \to b > 0 \) and \( I'(v_n) \to 0 \). We know what to do if there are only two intervals on which \( v_n \) spends an unbounded amount of time in \( B_{\eta/2}(K(V)) \). We now seek to reduce the general problem to exactly this case. To do this we need the following lemma:

**Lemma 2.1.19.** Suppose that \( I(v_n) \to b > 0 \) and \( I'(v_n) \to 0 \). Moreover, suppose that \((t_{n_j}, \bar{t}_{n_j})\) is an interval whose length \( \to \infty \) as \( n \to \infty \), and \( v_n(t) \in B_{\eta/2}(\xi) \) for some fixed \( \xi \in K(V) \), all \( t \in (t_{n_j}, \bar{t}_{n_j}) \) and all \( n \). Let \( t^j_n \) be any point in \((t_{n_j}, \bar{t}_{n_j})\) such that

(a) \( t^j_n - t_{n_j} \to \infty \) as \( n \to \infty \) and

(b) \( \bar{t}_{n_j} - t^j_n \to \infty \) as \( n \to \infty \).

Then \( \|v_n - \xi\|_{L^\infty[t^j_n - 1, t^j_n + 1]} \to 0 \).

**Proof.** Pick \( k_n \in \mathbb{Z} \) such that \( t^j_n - k_n \in [0, 1) \), and consider the sequence \( \tilde{v}_n(t) := v_n(t + k_n) \). Then we have \( I(\tilde{v}_n) = I(v_n) \), so \( I(\tilde{v}_n) \) is bounded. Because of the normalization provided by the choice of the \( k_n \), this implies that (on a subsequence) \( \tilde{v}_n \to v \) in \( \hat{E} \). We claim now that

(i) \( v \) satisfies (HS)

(ii) \( \| v - \xi \|_{L^\infty} \leq \frac{\eta}{2} \)

Assuming these, then Corollary 2.1.17 implies that \( v \equiv \xi \). But weak convergence in \( \hat{E} \) implies local uniform convergence, so

\[
\| \bar{v}_n - \xi \|_{L^\infty[-1,2]} \to 0. \tag{2.78}
\]

Now, if \( t_n^j - 1 \leq t \leq t_n^j + 1 \), then \( t_n^j - 1 - k_n \leq t - k_n \leq t_n^j + 1 - k_n \) hence \( t - k_n \in [-1,2] \). Therefore

\[
\| \bar{v}_n - \xi \|_{L^\infty[-1,2]} \geq \| \bar{v}_n - \xi \|_{L^\infty[t_n^j - k_n - 1, t_n^j - k_n + 1]} = \| v_n - \xi \|_{L^\infty[t_n^j - k_n - 1, t_n^j + 1]}, \tag{2.79}
\]

which is what we want.

To see (i), pick \( \varphi \in C^\infty_c(\mathbb{R}) \), and suppose that \( \text{supp}(\varphi) \subset (a,c) \). Taking \( \varphi_n(t) := \varphi(t + k_n) \), we have

\[
\int_\mathbb{R} \langle \dot{v}_n, \dot{\varphi} \rangle - \langle V_q(t, \bar{v}_n), \varphi \rangle dt = \int_a^c \langle \dot{v}_n(t - k_n), \dot{\varphi}(t) \rangle - \langle V_q(t, v_n(t - k_n)), \varphi(t) \rangle dt = \int_{a-k_n}^{c-k_n} \langle \dot{v}_n(t), \dot{\varphi}(t + k_n) \rangle - \langle V_q(t + k_n, v_n(t)), \varphi(t + k_n) \rangle dt = \int_{a-k_n}^{c-k_n} \langle \dot{v}_n(t), \varphi(t + k_n) \rangle - \langle V_q(t, v_n(t)), \varphi(t + k_n) \rangle dt = I'(v_n)\varphi_n \to 0
\]

since \( \varphi_n \) is bounded in \( W^{1,2} \).

But

\[
\int_\mathbb{R} \langle \dot{v}_n, \dot{\varphi} \rangle dt \to \int_\mathbb{R} \langle \dot{v}, \dot{\varphi} \rangle dt
\]
by the weak convergence in $\hat{E}$. Because $\varphi$ has compact support, we also have

$$\int_{\mathbb{R}} \langle V_q(t, \tilde{v}_n), \varphi \rangle dt \to \int_{\mathbb{R}} \langle V_q(t, v), \varphi \rangle dt.$$ 

Thus, we must have

$$\int_{\mathbb{R}} \langle \dot{v}, \dot{\varphi} \rangle - \langle V_q(t, v), \varphi \rangle dt = 0$$

for all $\varphi \in C_0^\infty$, hence $v$ satisfies (HS).

It remains only to prove that (ii) is satisfied. Note that $v_n(t) \in B_{\eta/2}(\xi)$ for $t \in (t_{n_j}, \bar{t}_{n_j})$. Since $v$ is the pointwise limit of the $\tilde{v}_n$, in order to show (ii), it suffices to show that for a fixed $\hat{t}$, we have $\tilde{v}_n(\hat{t}) \in B_{\eta/2}(\xi)$ for all large $n$. (How large $n$ must be might depend on $\hat{t}$.) Because $\tilde{v}_n(\hat{t}) = v_n(\hat{t} + k_n)$, it suffices to show that $\hat{t} + k_n \in (t_{n_j}, \bar{t}_{n_j})$ for all large $n$. Equivalently, we need to show that $\hat{t} \in (t_{n_j} - k_n, \bar{t}_{n_j} - k_n)$ for all large $n$. We claim that we have

$$t_{n_j} - k_n \to -\infty \quad \text{and} \quad \bar{t}_{n_j} - k_n \to \infty,$$  

(2.81)

which would then imply our claim. Notice that

$$t_{n_j} - k_n = (t_{n_j} - t^j_n) - (t^j_n - k_n).$$  

(2.82)

But the second term on the right in (2.82) is bounded by the definition of $k_n$, and by (a) of our assumptions, $(t_{n_j} - t^j_n) \to -\infty$ as $n \to \infty$. Thus (2.82) implies that $t_{n_j} - k_n \to -\infty$ as $n \to \infty$. Similarly,

$$\bar{t}_{n_j} - k_n = (\bar{t}_{n_j} - t^j_n) - (t^j_n - k_n).$$  

(2.83)

The second term on the right in (2.83) is bounded by the definition of $k_n$ while the first tends to $\infty$ as $n \to \infty$ by assumption (b). Therefore, (2.83) implies that $\bar{t}_{n_j} - k_n \to \infty$ as $n \to \infty$, and (2.81) is proven. \qed
With these preliminaries out of the way, we can finally turn to the general case when there are more than two intervals where \( v_n \in B_{\eta/2}(K(V)) \). We know what to do if there are only two intervals on which \( v_n \) spends an increasing amount of time in \( B_{\eta/2}(K(V)) \). Suppose that the only intervals whose lengths \( \to \infty \) as \( n \to \infty \) and in which \( v_n(t) \in B_{\eta/2}(K(V)) \) are

\[ (-\infty, \bar{t}_{n_1}), (t_{n_2}, \bar{t}_{n_2}), (t_{n_3}, \bar{t}_{n_3}), \ldots, (\bar{t}_{l(n)}, \infty). \]

By passing to a subsequence if necessary, we may assume that \( l(n) \) is independent of \( n \). Call this constant value \( l \). We must have \( l \in \mathbb{N} \) and \( l \geq 2 \). Since we have already dealt with the case of \( l = 2 \), suppose \( l > 2 \). Let us suppose now that

1. \( v_n(-\infty) = 0 \) for all \( n \)
2. \( v_n(\infty) = \xi_l \) for all \( n \)

By passing to a subsequence, we may also assume that

1. \( v_n(t) \in B_{\eta/2}(\xi_2) \) for all \( t \in (\bar{t}_{n_2}, \bar{t}_{n_2}) \) and all \( n \).
2. \( v_n(t) \in B_{\eta/2}(\xi_3) \) for all \( t \in (\bar{t}_{n_3}, \bar{t}_{n_3}) \)
3. \( v_n(t) \in B_{\eta/2}(\xi_j) \) for all \( t \in (\bar{t}_{n_j}, \bar{t}_{n_j}) \) for \( j \leq l \).

The content of the next theorem is that we may split the sequence \( v_n \) into \( l - 1 \) Palais-Smale sequences such that on each one, there are only two intervals where an unbounded amount of time is spent in \( B_{\eta/2}(K(V)) \).
Proposition 2.1.20. Let $v_n$ be as above. Then there are PS sequences $v^i_n$ for $i = 1, 2, \ldots, l - 1$ such that

(i) $v^i_n(t) \in B_{\eta/2}(\xi_i)$ for $t < \bar{t}_n$

(ii) $v^i_n(t) \in B_{\eta/2}(\xi_{i+1})$ for $t > t_{n+1}$

(iii) $(-\infty, \bar{t}_n)$ and $(t_{n+1}, \infty)$ are the only intervals with unbounded length where $v^i_n(t) \in B_{\eta/2}(K(V))$

(iv) $|I(v^i_n) - \int_{t_n}^{t_{n+1}} \left( \frac{1}{2} |\dot{v}_n|^2 - V(t, v_n) \right) dt| \to 0$ as $n \to \infty$ for any $i = 1, 2, \ldots, l - 1$.

Proof. For $i = 2, 3, \ldots, l - 2$, pick points $t^i_n \in (\bar{t}_n, \tilde{t}_n)$ such that assumptions (a) and (b) of Lemma 2.1.19 hold. (For example, we could take $t^i_n$ to be the midpoint of $(\bar{t}_n, \tilde{t}_n)$.) Then, let

$$v^i_n(t) := \begin{cases} 
\xi_i & \text{for } t < t^i_n - 1 \\
(t - t^i_n + 1)(\xi_i - v_n(t^i_n)) + \xi_i & \text{for } t^i_n - 1 \leq t \leq t^i_n \\
v_n(t) & \text{for } t^i_n < t < t^{i+1}_n \\
(t^{i+1}_n + 1 - t)(v_n(t^{i+1}_n) - \xi_{i+1}) - \xi_{i+1} & \text{for } t^i_{n+1} \leq t \leq t^{i+1}_n + 1 \\
\xi_{i+1} & \text{for } t > t^{i+1}_n + 1 
\end{cases} \tag{2.84}$$

For $i = 1$, we have $t^1_n = -\infty$ and we take $v^1_n(t) = v_n(t)$ for $t < t^2_n$. For $t^2_n \leq t \leq t^2_n + 1$, we interpolate as above between $v_n(t^2_n)$ and $\xi_2$, and then take $v^1_n(t) = \xi_2$ for all $t > t^2_n + 1$. For $i = l - 1$, we have $t^l_n = \infty$, and so we take $v^{l-1}_n(t) = v_n(t)$ for $t > t^{l-1}_n$. For $t^{l-1}_n \leq t \leq t^{l-1}_n$, we interpolate between $v_n(t^{l-1}_n)$ and $\xi_{l-1}$, while for $t < t^{l-1}_n$, $v^{l-1}_n(t) = \xi_{l-1}$. In any case, the arguments that follow are the same, so we ignore these two special cases. Notice that we have obtained $v^i_n$ from $v_n$ by linear interpolation at $t^i_n, t^{i+1}_n$ between $v_n$ and $\xi_i, \xi_{i+1}$, respectively. In addition, for each
$n$, the chain $\{v^1_n, v^2_n, \ldots, v^{l-1}_n\}$ connects 0 to $\xi_l$, and “shadows” $v_n$ in the sense that $v^n_i$ is $W^{1,2}$ close to $v_n$ for long intervals.

For future reference, note that if $\|v_n\|_{L^\infty(\mathbb{R})} \leq C$, then we must have $\|v^n_i\|_{L^\infty(\mathbb{R})} \leq C + \eta$. To see this, note that for $t \in (t^i_n, t^{i+1}_n)$, we have $v^n_i(t) = v_n(t)$, hence $\|v^n_i\|_{L^\infty(t^i_n, t^{i+1}_n)} \leq C$. If $t \leq t^i_n$, then $v^n_i(t) \in B_{\eta/2}(\xi_i)$. But

$$|v^n_i(t)| \leq |v^n_i(t) - v_n(t^i_n)| + |v_n(t^i_n)| \leq |v^n_i(t) - \xi_i| + |\xi_i - v_n(t^i_n)| + \|v_n\|_{L^\infty} \leq \frac{\eta}{2} + \frac{\eta}{2} + C = C + \eta. \quad (2.85)$$

The case for $t \geq t^{i+1}_n$ is similar. We now verify (i)-(iv).

(i): Suppose that $t < t^i_n$. If $t \leq t^i_n$, then (i) is obvious, since we have $v^n_i(t) = \xi_i$ or $v^n_i(t)$ is a convex combination of elements in $B_{\eta/2}(\xi_i)$. If $t^i_n < t < t^i_n$, then $v^n_i(t) = v_n(t)$, and on this interval $v_n(t) \in B_{\eta/2}(\xi_i)$.

(ii): Suppose that $t > t^{i+1}_n$. If $t \geq t^{i+1}_n$, then $v^n_i(t) = \xi_{i+1}$ or $v^n_i(t)$ is a convex combination of elements in $B_{\eta/2}(\xi_{i+1})$. If $t^{i+1}_n < t < t^{i+1}_n$, then $v^n_i(t) = v_n(t)$ and on this interval $v_n(t) \in B_{\eta/2}(\xi_{i+1})$.

(iii): Suppose that there is another interval $t^a_n, t^c_n$ such that $t^c_n - t^a_n \to \infty$ as $n \to \infty$ and on which $v^n_i(t) \in B_{\eta/2}(K(2))$. Then $(t^a_n, t^b_n) \subset (t^c_n, t^{i+1}_n)$. But by Lemma 2.1.12, $M \geq t^{i+1}_n - \tilde{t}^{i+1}_n \geq t^b_n - t^a_n \to \infty$, which is impossible. Thus there can be no such interval.
(iv): Notice that we have
\[ 0 \leq I(v^n_i) - \int_{t^n_{i-1}}^{t^n_{i+1}} \left( \frac{1}{2} |\dot{v}^i_n|^2 - V(t, v_n) \right) dt \]
\[ = \int_{t^n_{i-1}}^{t^n_{i}} \left( \frac{1}{2} |\dot{v}^i_n|^2 - V(t, v^n_i) \right) dt + \int_{t^n_{i}}^{t^n_{i+1}} \left( \frac{1}{2} |\dot{v}^i_n|^2 - V(t, v^n_i) \right) dt \]
\[ = \int_{t^n_{i-1}}^{t^n_{i}} \left( \frac{1}{2} |\xi_i - v_n(t^n_i)|^2 - V(t, \text{linear in } t \text{ piece}) \right) dt \]
\[ + \int_{t^n_{i}}^{t^n_{i+1}} \left( \frac{1}{2} |\xi_{i+1} - v_n(t^n_{i+1})|^2 - V(t, \text{linear in } t \text{ piece}) \right) dt \]
\[ = \frac{1}{2} |\xi_i - v_n(t^n_i)|^2 + \frac{1}{2} |\xi_{i+1} - v_n(t^n_{i+1})|^2 + \int_{t^n_{i-1}}^{t^n_{i}} -V(t, \text{linear in } t \text{ piece})dt \]
\[ + \int_{t^n_{i}}^{t^n_{i+1}} -V(t, \text{linear in } t \text{ piece})dt. \]

Since \( v_n(t^n_i) \to \xi_i \) and \( v_n(t^n_{i+1}) \to \xi_{i+1} \) by Lemma 2.1.19, all the terms on the right in (2.87) go to 0 as \( n \to \infty \).

It remains only to see that the sequences \( v^n_i \) are PS sequences. Since \( I(v^n_i) \) is bounded, it remains only to see that \( I'(v^n_i) \to 0 \) as \( n \to \infty \) for \( i = 1, 2, \ldots, l - 1 \).

Let \( \varphi \in W^{1,2} \), and suppose that \( \| \varphi \|_{W^{1,2}} \leq 1 \). We want to show that \( |I'(v^n_i)\varphi| \to 0 \) independently of such \( \varphi \). We have
\[ I'(v^n_i)\varphi = \int_{t^n_{i-1}}^{t^n_{i}} \left( \langle \xi_i - v_n(t^n_i), \varphi \rangle - \langle V_q(t, v_n), \varphi \rangle \right) dt \]
\[ + \int_{t^n_{i}}^{t^n_{i+1}} \langle \dot{v}_n, \varphi \rangle - \langle V_q(t, v_n), \varphi \rangle dt \]
\[ + \int_{t^n_{i}}^{t^n_{i+1}} \langle \xi_{i+1} - v_n(t^n_{i+1}), \varphi \rangle - \langle V_q(t, \text{linear in } t \text{ piece}), \varphi \rangle dt. \]

The second two terms in (2.88) \( \to 0 \) as \( n \to \infty \) independently of \( \varphi \) because:
\[ \left| \int_{t^n_{i-1}}^{t^n_{i}} \left( \langle \xi_i - v_n(t^n_i), \varphi \rangle \right) dt \right| \leq |\xi_i - v_n(t^n_i)| \| \varphi \|_{L^2(t^n_{i-1}, t^n_{i})} \]
\[ \leq |\xi_i - v_n(t^n_i)| \]
and

$$\left| \int_{t_{n-1}}^{t_n} \langle V_q(t, \text{linear in } t \text{ piece}), \varphi \rangle dt \right| \leq \| V_q(t, \text{linear in } t \text{ piece}) \|_{L^\infty(t_n^{-1}, t_n)} \| \varphi \|_{L^2}$$

(2.90)

$$\leq \| V_q(t, \text{linear in } t \text{ piece}) \|_{L^\infty(t_n^{-1}, t_n^+)}.$$ 

Both (2.89) and (2.90) go to 0 as \(n \to \infty\), which follows from the uniform convergence of the linear piece to 0. This in turn follows from Lemma 2.1.19.

To finish the proof, it remains to verify that

$$\int_{t_n}^{t_{n+1}} \langle \dot{\varphi}_n, \dot{\varphi} \rangle - \langle V_q(t, v_n), \varphi \rangle dt \to 0$$

(2.91)

as \(n \to \infty\), independently of \(\varphi\).

For any \(\varphi \in W^{1,2}\), we define \(\varphi_n\) by:

$$\varphi_n(t) := \begin{cases} 
0 & \text{for } t < t_n^i - 1 \\
(t - t_n^i + 1) \varphi(t_n^i) & \text{for } t_n^i - 1 \leq t \leq t_n^i \\
\varphi(t) & \text{for } t_n^i < t < t_n^{i+1} \\
(t_n^{i+1} + 1 - t) \varphi(t_n^{i+1}) & \text{for } t_n^{i+1} \leq t \leq t_n^{i+1} + 1 \\
0 & \text{for } t > t_n^{i+1}
\end{cases}$$

Thus, \(\varphi_n\) is obtained from \(\varphi\) by linear interpolation between \(\varphi\) and 0 at \(t_n^i\) and \(t_n^{i+1}\). We claim that \(\varphi_n\) is bounded in \(W^{1,2}\) independently of \(n\) and \(\varphi\). To see this, we have: (using the fact that \(\| \varphi \|_{L^\infty(\mathbb{R})} \leq C \| \varphi \|_{W^{1,2}(\mathbb{R})}\), see [4])

$$\| \varphi_n \|_{L^2}^2 \leq \| \varphi \|_{L^2}^2 + 2 \| \varphi \|_{L^\infty}^2 \leq (2C^2 + 1) \| \varphi \|_{W^{1,2}}^2 \leq 2C^2 + 1$$
and

$$\| \dot{\varphi}_n \|^2_{L^2} \leq \| \dot{\varphi} \|^2_{L^2} + 2\| \varphi \|^2_{L^\infty} \leq (2C^2 + 1)\| \varphi \|^2_{W^{1,2}} \leq 2C^2 + 1$$

Therefore, \( |I'(v_n)\varphi_n| \leq K \| I'(v_n) \| \to 0 \) as \( n \to \infty \) uniformly for \( \| \varphi \|_{W^{1,2}} \leq 1 \). We have

\[
I'(v^i_n)\varphi - I'(v_n)\varphi_n = \int_{t_{i-1}^n}^{t_i^n} (\xi_i - v_n(t_i^n), \dot{\varphi}) - (V_q(t, \text{linear in } t \text{ piece}), \varphi) \, dt \\
+ \int_{t_{i-1}^n}^{t_{i+1}^n} (\xi_{i+1} - v_n(t_{i+1}^n), \dot{\varphi}) - (V_q(t, \text{linear in } t \text{ piece}), \varphi) \, dt \\
- \int_{t_{i-1}^n}^{t_i^n} \langle \dot{v}_n, \varphi(t_i^n) \rangle - \langle V_q(t, v_n), (t - t_i^n + 1)\varphi(t_i^n) \rangle \, dt \\
- \int_{t_{i+1}^n}^{t_{i+1}^{i+1}} \langle \dot{v}_n, \varphi(t_{i+1}^{i+1}) \rangle - \langle V_q(t, v_n), (t_{i+1}^{i+1} + 1 - t)\varphi(t_{i+1}^{i+1}) \rangle \, dt \tag{2.92}
\]

We already know that the first two terms in (2.92) go to 0 as \( n \to \infty \) independently of \( \varphi \), so we turn our attention to showing this for the second two. This will be a consequence of Lemma 2.1.19, which implies \( \| v_n - \xi \|_{L^\infty[t_{i-1}^n, t_i^n]} \to 0 \), hence \( V_q(t, v_n(t)) \to 0 \) uniformly on \([t_{i-1}^n, t_i^n]\) as \( n \to \infty \). Therefore,

\[
\left| \int_{t_{i-1}^n}^{t_i^n} \langle V_q(t, v_n), (t - t_i^n + 1)\varphi(t_i^n) \rangle \, dt \right| \leq \| \varphi \|_{L^\infty} \| V_q(t, v_n) \|_{L^\infty[t_{i-1}^n, t_i^n]} \leq C\| V_q(t, v_n) \|_{L^\infty[t_{i-1}^n, t_i^n]} \to 0 \tag{2.93}
\]

as \( n \to \infty \). The

\[
\int_{t_{i+1}^n}^{t_{i+1}^{i+1}} \langle V_q(t, v_n), (t_{i+1}^{i+1} + 1 - t)\varphi(t_{i+1}^{i+1}) \rangle \, dt
\]

term in (2.92) is dealt with similarly. Next, we turn our attention to the
\[
\int_{t_{n-1}}^{t_n} \langle \dot{v}_n, \varphi(t_n^i) \rangle \, dt \text{ term in (2.92) We have the following:}
\]
\[
\left| \int_{t_{n-1}}^{t_n} \langle \dot{v}_n, \varphi(t_n^i) \rangle \, dt \right| = \left| \sum_{k=1}^{d} \left( \varphi_k(t_n^i) \int_{t_{n-1}}^{t_n} \dot{v}_n^k(t) \, dt \right) \right| \tag{2.94}
\]
where \( \varphi_k(t), v_n^k(t) \) are the components of \( v(t), \varphi(t) \in \mathbb{R}^d \) for \( k = 1, 2, \ldots, d \). Therefore, (2.94) implies that
\[
\left| \int_{t_{n-1}}^{t_n} \langle \dot{v}_n, \varphi(t_n^i) \rangle \, dt \right| = \left| \sum_{k=1}^{d} \left( \varphi_k(t_n^i) \left( v_n^k(t_n^i) - v_n^k(t_{n-1}^i) \right) \right) \right| \tag{2.95}
\]
because of Lemma 2.1.19. We deal with the \( \int_{t_{n+1}^i}^{t_{n+1}^i+1} \langle \dot{v}_n, \varphi(t_n^i+1) \rangle \) term similarly.

Altogether, we have
\[
I'(v_n^i) \varphi = \text{ a sum of terms that } \to 0 \text{ as } n \to \infty, \text{ independent of } \varphi
\]
and so \( I'(v_n^i) \to 0 \) as \( n \to \infty \), which concludes the proof. \( \square \)

Suppose then that we have a sequence \( v_n \in \hat{E} \) such that \( I(v_n) \to b > 0 \) and \( I'(v_n) \to 0 \). Moreover, suppose that \( v_n(-\infty) = 0 \) and \( v_n(\infty) = \xi \neq 0 \) for all \( n \).

By Lemma 2.1.3, we know that \( v_n \) is bounded in \( L^\infty(\mathbb{R}) \). We also know that there is a finite number of intervals, say \( l \), whose lengths are unbounded in \( n \) and on which \( v_n(t) \in B_{\eta/2}(\tilde{\xi}_j) \) for some \( \tilde{\xi}_j \in K(V) \), \( j = 1, 2, \ldots, l \). By Proposition 2.1.20, there are \( l - 1 \) PS sequences \( v_n^i \in \hat{E} \) such that each one has only two intervals whose lengths are unbounded in \( n \) and on which \( v_n^i(t) \in B_{\eta/2}(K(V)) \). Moreover, we can pick a subsequence of \( v_n \) such that the \( v_n^i \) form a chain from 0 to \( \xi \). Notice
that by (iv) of Proposition 2.1.20 and the \( L^\infty \) boundedness of \( v_n \), we know \( I(v_n^i) \) is bounded. Passing to subsequences yet again, we may assume there exist \( b_i > 0 \) such that for \( i = 1, 2, \ldots, l - 1 \), \( I(v_n^i) \to b_i \). By Proposition 2.1.13, for each sequence \( v_n^i \), there is a sequence \( k_n^i \in \mathbb{Z} \) such that \( \| \tau_{k_n^i} v_n^i - v^i \|_{W^{1,2}} \to 0 \), where \( v^i \) is a critical point of \( I \) such that \( I(v^i) = b_i \). How is \( \sum_{i=1}^{l-1} b_i \) related to \( b \)?

We claim \( \sum_{i=1}^{l-1} b_i = b \). To see this, note that (since \( t_1 = -\infty \) and \( \bar{t}_l = +\infty \))

\[
I(v_n) = \sum_{i=1}^{l-1} \int_{t_n^i}^{t_n^{i+1}} \left( \frac{1}{2} |\dot{v}_n|^2 - V(t, v_n) \right) dt. \tag{2.96}
\]

By (iv) of Proposition 2.1.20

\[
I(v_n^i) - \int_{t_n^i}^{t_n^{i+1}} \left( \frac{1}{2} |\dot{v}_n|^2 - V(t, v_n) \right) dt \to 0 \tag{2.97}
\]
as \( n \to \infty \) for \( i = 1, 2, \ldots, l - 1 \). Thus, combining (2.96) and (2.97), we have

\[
\left| b - \sum_{i=1}^{l-1} b_i \right| \leq |b - I(v_n)| + \left| I(v_n) - \sum_{i=1}^{l-1} I(v_n^i) \right| + \left| \sum_{i=1}^{l-1} (I(v_n^i) - b_i) \right|
\]

\[
= |b - I(v_n)| + \sum_{i=1}^{l-1} \left( \left( \int_{t_n^i}^{t_n^{i+1}} \frac{1}{2} |\dot{v}_n|^2 - V(t, v_n) \ dt \right) - I(v_n^i) \right) \tag{2.98}
\]

\[
+ \sum_{i=1}^{l-1} (I(v_n^i) - b_i).
\]

Letting \( n \to \infty \) then implies that \( \sum_{i=1}^{l-1} b_i = b \).

Altogether, we have proven the following result on the structure of (PS) sequences:

**Theorem 2.1.21.** If \( v_n \in \hat{E} \) is a sequence such that

\((C1)\ v_n(-\infty) = 0 \) and \( v_n(\infty) = \tilde{\xi} \) for some fixed \( \tilde{\xi} \in K(V) \) and all \( n \)
(C2) $I(v_n) \to b > 0$ and $I'(v_n) \to 0$.

Then, there exists a finite set of solutions to (HS) $v^1, v^2, \ldots, v^m$ such

(i) $\sum_{i=1}^{m} I(v^i) = b$

(ii) $v^1(-\infty) = 0$, $v^m(\infty) = \xi$, and $v^i(\infty) = v^{i+1}(-\infty) \in K(V)$ for every $i = 1, 2, \ldots, m - 1$.

Here $m + 1$ is the number of intervals with unbounded length in $n$ on which $v_n(t) \in B_{\eta/2}(K(V))$.

Remark: (1) If we have a sequence $v_n$ such that $I(v_n) \to b > 0$ and $I'(v_n) \to 0$, then by Lemma 2.1.3, we know that $\|v_n\|_{L^\infty(\mathbb{R})} \leq C$. But then $\|v^i_n\|_{L^\infty} \leq C + \eta$.

By the $W^{1,2}$ convergence of $\tau_{k_n} v^i_n$ to $v^i$ from Proposition 2.1.13, we know that in fact $\tau_{k_n} v^i_n$ converges to $v^i$ in $L^\infty$, hence $\|v^i\|_{L^\infty} \leq C + \eta$.

(2) It is possible to give a precise convergence statement about the manner in which $v_n$ converges to a “chain” of solutions of (HS). Such a statement would be very similar to Proposition 3.10 in [19]. However, we find it more convenient to use Proposition 2.1.20 to break $v_n$ into $m$ (PS) sequences $v^i_n$, and then apply Proposition 2.1.13 to each $v^i_n$.

Observe also that we have not excluded the case of homoclinic solutions of (HS) in Theorem 2.1.21.

2.2 Mountain Pass Points

Finally, we can prove the existence of solutions of (HS) that are not minimizers. Let us consider the case when $K(V) = \{0, \xi\}$. It is known that there exist heteroclinic
connections between the two equilibria by minimizing $I$ over an appropriate subset of $\hat{E}$ (see [19], [22]). Suppose that $q$ is a heteroclinic going from $0$ to $\xi$. Notice that by (V1), $\tilde{q}(t) := q(t + 1)$ is also a minimizer, and $I(\tilde{q}) = I(q)$. Once we have two minimizers, we can ask ourselves if there is a critical point of mountain pass type “between” them. To do this, we make the following assumption about $I$ and $q$.

$q$ is an isolated minimizer of $I$

For example, the assumptions about the potential $V$ imply that $I$ is actually $C^2$, so if $I''(q)(\varphi, \varphi) \geq a\|\varphi\|^2$ for some $a > 0$, then the assumption above would be satisfied.

Let $\tilde{J}(u) := J_q(u) - I(q)$. Notice that by Proposition 2.1.6, $\tilde{J} \in C^1(E, \mathbb{R})$ and $\tilde{J}(u) \geq 0$ for all $u \in E$. Let

$$c := \inf_{h \in \Gamma} \max_{0 \leq s \leq 1} \tilde{J}(h_s)$$

where $\Gamma := \{h \in C([0, 1], E) \mid h(0) \equiv 0$ and $h(1)(t) = q(t + 1) - q(t)\}$. Notice that by our assumption about $q$ being an isolated local minimizer, $c > 0$. In fact, we have the following proposition:

**Proposition 2.2.1.** If $c = 0$, then for every $r \in (0, \|q(t + 1) - q(t)\|_{W^{1,2}})$, there is a $k = k(r) \in \mathbb{Z}$ and a solution $\tilde{q}$ of (HS) such that $I(\tilde{q}) = I(q)$ and $\|\tilde{q} - \tau_kq\|_{W^{1,2}} = r$.

**Proof.** If $c = 0$, there is a sequence $h_n \in \Gamma$ such that $\max_{0 \leq s \leq 1} \tilde{J}(h(s)) < 1/n$. For any $r \in (0, \|q(t + 1) - q(t)\|_{W^{1,2}})$, let $s_{n,r}$ be the smallest $s$ for which $\|h_n(s)\|_{W^{1,2}} = r$. Then $\tilde{J}(h_n(s_{n,r})) < 1/n$, so $h_n(s_{n,r})$ is a minimizing sequence for $\tilde{J}$. We next apply Ekeland’s principle: for any $n$ and $u \in E$ with $\tilde{J}(u) < \inf_{x \in E} \tilde{J}(x) + 1/n$, there is a $v \in E$ such that...
(a) $\tilde{J}(v) \leq \tilde{J}(u)$

(b) $\|u - v\|_E \leq 1/\sqrt{n}$

(c) $\|\tilde{J}'(v)\|_{E'} \leq 1/\sqrt{n}$.

Taking $u = h_n(s_{n,r})$, we have a sequence $v_n \in E$ such that

(a') $\tilde{J}(v_n) \leq 1/n$

(b') $\|h_n(s_{n,r}) - v_n\|_E \leq 1/\sqrt{n}$

(c') $\|\tilde{J}'(v_n)\|_{E'} \leq 1/\sqrt{n}$.

Notice that (b') implies that $\|v_n\|_E \to r$ as $n \to \infty$. Taking

$$w_n := q + v_n \in \hat{E}, \quad (2.100)$$

the definition of $\tilde{J}(u)$, (a') and (c') imply that

$I(w_n) \to I(q)$ and $\|I'(w_n)\| \to 0$

as $n \to \infty$. In addition, each $w_n$ has the same asymptotics: $w_n(-\infty) = 0$ and $w_n(\infty) = \xi$ for all $n$. We claim now that there are only two intervals $(-\infty, \bar{t}_n)$ and $(\underline{t}_n, \infty)$ with unbounded lengths in $n$ where $w_n(t) \in B_{\eta/2}(K(V))$. If not, then there are at least three such intervals, and so by Theorem 2.1.21, we have

$$I(q) = \sum_{i=1}^{m} I(x^i) \quad (2.101)$$

for some $m \geq 2$ and the $x^i$ are heteroclinic or homoclinic solutions of (HS). Now, at least one of these $x^i$ connects 0 to $\xi$. For simplicity, suppose that $x^1$ connects 0
to $\xi$. Thus (2.101) implies that

$$0 \geq I(q) - I(x^i) = \sum_{i=1}^{m-1} I(x^i) \geq (m - 1)\nu > 0.$$  

(2.102)

Thus, $m = 1$ and so there are only two such intervals. Then, we may apply Proposition 2.1.13: there exist $k_n \in \mathbb{Z}$ and a solution $\tilde{q}$ of (HS) heteroclinic from 0 to $\xi$ such that

$$\|\tau_{k_n} w_n - \tilde{q}\|_{W^{1,2}} \to 0$$  

(2.103)

as $n \to \infty$. By (2.100), (2.103) and (b') imply that

$$\left|\|\tau_{k_n} q - \tilde{q}\|_{W^{1,2}} - r\right| \leq \|\tau_{k_n} q - \tilde{q}\|_{W^{1,2}} - \|\tau_{k_n} v_n\|_{W^{1,2}} + \|\tau_{k_n} v_n\|_{W^{1,2}} - r \right|$$  

(2.104)

$$\leq \|\tau_{k_n} (q + v_n) - \tilde{q}\|_{W^{1,2}} + \|\tau_{k_n} v_n\|_{W^{1,2}} - r \right| \to 0$$

as $n \to \infty$. Thus, $\|\tau_{k_n} q - \tilde{q}\|_{W^{1,2}} \to r$ as $n \to \infty$. We claim now that if $\{k_n\} \subset \mathbb{Z}$ is unbounded, then $\|\tau_{k_n} q - \tilde{q}\|_{W^{1,2}} \to \infty$. To see this, if $\{k_n\}$ is not bounded, then passing to a subsequence, we have either $k_n \to -\infty$ or $k_n \to \infty$ as $n \to \infty$. Since $q(-\infty) = 0 = \tilde{q}(-\infty)$ and $q(\infty) = \xi = \tilde{q}(\infty)$, there are constants $\underline{a} < \bar{a}$ such that for $t < \underline{a}$, $q(t), \tilde{q}(t) \in B_{1/4}(0)$ and for $t > \bar{a}$, $q(t), \tilde{q}(t) \in B_{1/4}(\xi)$. Suppose now that $k_n \to \infty$. Then, for all $t > \bar{a}$, $\tilde{q}(t) \in B_{1/4}(\xi)$. On the other hand, for all $t < \underline{a} + k_n$, we have $\tau_{k_n} q(t) \in B_{1/4}(0)$. Thus, for all $t \in [\underline{a}, \underline{a} + k_n]$, we must have $\tau_{k_n} q(t) - \tilde{q}(t) \in B_{1/2}(-\xi)$, and so

$$\|\tau_{k_n} q - \tilde{q}\|_{W^{1,2}}^2 \geq \int_{\underline{a}}^{\underline{a} + k_n} |\tau_{k_n} q(t) - \tilde{q}|^2 dt$$

$$\geq (\underline{a} + k_n - \bar{a})|\xi - 1/2|^2 \to \infty$$
as \( n \to \infty \). A similar proof holds if \( k_n \to -\infty \) as \( n \to \infty \). Thus, \( k_n = k^* \) for some fixed \( k^* \) and all large \( n \). But then (2.104) implies that

\[
\|\tau_{k^*}q - \tilde{q}\|_{W^{1,2}} = \|q - \tau_{-k^*}\tilde{q}\|_{W^{1,2}} = r.
\] (2.105)

Since this is true for any \( r \in (0, \|q(t + 1) - q(t)\|_{W^{1,2}}) \), the proof is complete. \( \square \)

In other words, if \( c = 0 \), then there is a sequence \( q_n \) of heteroclinic solutions of (HS) connecting 0 to \( \xi \) such that \( \|q_n - q\|_{W^{1,2}} \to 0 \), which would contradict the assumption that \( q \) is an isolated minimizer of \( I \).

**Theorem 2.2.2.** If \( c < \nu \) (where \( \nu \) is from Corollary 2.1.18), then there is a \( u \in E \) such that \( \tilde{J}(u) = c \) and \( \tilde{J}'(u) = 0 \).

Because \( \tilde{J}'(u) = 0 \), we have \( 0 = J'_q(u) \), hence \( u + q \) is a solution of (HS), which goes from 0 to \( \xi \). Moreover, \( I(u + q) - I(q) = c > 0 \), so \( u + q \neq q \), and we have a heteroclinic from 0 to \( \xi \) that is geometrically distinct from \( q \). We shall give an example later of when this theorem is satisfied.

**Proof.** Following [25], we can construct a Palais-Smale sequence \( \{u_n\} \subset E \) such that \( \tilde{J}(u_n) \to c \) and \( \tilde{J}'(u_n) \to 0 \). Using Willem’s notation from [25], we set \( M := [0,1], M_0 := \{0,1\} \). Then our \( \Gamma \) is the same as that in Theorem 2.8 of [25], and since \( c > 0 = \tilde{J}(h(0)) = \tilde{J}(h(1)) \), the assumptions of that theorem are satisfied. Thus, by Theorem 2.9 of [25], there is such a Palais-Smale sequence.

Let \( v_n := u_n + q \). Then, we have \( I(v_n) - I(q) \to c \), hence \( I(v_n) \to c + I(q) \). Notice that we also have \( I'(v_n) \to 0 \), and \( v_n(-\infty) = 0, v_n(\infty) = \xi \) for all \( n \). By Theorem 2.1.21, there exists an \( m \in \mathbb{N} \) such that \( c + I(q) = \sum_{i=1}^{m} I(v^i) \), where the \( v^i \) are heteroclinic or homoclinic solutions of (HS). Because \( v^i \) is a chain that connects
0 to $\xi$, there is at least one $v^i$ that is heteroclinic from 0 to $\xi$. For notational convenience, let this heteroclinic be $v^1$. We claim now that $v^1$ is the only term in $\sum_{i=1}^m I(v^i)$, i.e. $m = 1$. If not, we have $\nu > c > I(v^1) - I(q) + \sum_{i=1}^{m-1} I(v^i) \geq \nu$ since $I(v^1) - I(q) \geq 0$ and $m \geq 2$. Thus, we must have $c + I(q) = I(v^1)$. Then, if we define $u := v^1 - q$, we have

$$J(u) = J_q(u) - I(q) = I(u + q) - I(q) = I(v^1) - I(q) = c$$

Moreover, $u + q = v^1$ is a solution to (HS), so $J'(u) = 0$.

What if $c \geq \nu$? Then we do not have such precise information as above. However, we have the following

**Theorem 2.2.3.** Let $q$ be a minimal heteroclinic solution of (HS), connecting 0 to $\xi$, and $p$ be a minimal heteroclinic solution of (HS), connecting $\xi$ to 0. If $c \neq k_1 I(q) + k_2 I(p)$ for $k_1, k_2 \in \mathbb{N}, k_1, k_2 \geq 0$ then there is a non-constant $v$ with $I(v) < \infty, I'(v) = 0, v(\pm \infty) \in K(V)$ and $v \not\equiv p, v \not\equiv q$.

**Proof.** Suppose that there is no such $v$. We can find a sequence $\{u_n\} \subset E$ such that $J(u_n) \to c$ and $J'(u_n) \to 0$. If we define $v_n := u_n + q$, we have $J(u_n) = I(u_n + q) = I(v_n) \to c$ and $I'(u_n) \to 0$. Thus, by Theorem 2.1.21, there is a finite chain $v^1, v^2, \ldots, v^l$ of critical points of $I$ such that $c + I(q) = \sum_{i=1}^l I(v^i)$. Now, by our assumption, each $v^i$ is either $p$ or $q$. Notice that there is at least one $q$ in the sum, since $v^i$ is a chain that connects 0 and $\xi$.

$$c + I(q) = \sum_{j=1}^k I(q) + \sum_{j=1}^{l-k} I(p),$$
hence
\[ c = \sum_{j=1}^{k-1} I(q) + \sum_{j=1}^{l-k} I(p) \]
which is a contradiction. \(\square\)

**Remark:** (1) \(v\) is either a heteroclinic, or a homoclinic solution of (HS).

(2) We suspect that the condition \(c \neq k_1 I(q) + k_2 I(p)\) is generic, in the sense that it should be possible to find potentials \(V_n\) for which this condition is true, and \(V_n \to V\) as \(n \to \infty\). However, we are not at present able to prove this.

If we know about critical values corresponding to homoclinic solutions, we can sharpen the previous result:

**Corollary 2.2.4.** Under the conditions of the preceding theorem, if we also know that \(c \neq k_1 I(q) + k_2 I(p) + \sum_{\text{finite}} I(h_j)\) where the \(h_j\) are homoclinic solutions to (HS), then there is a heteroclinic solution connecting 0 and \(\xi\) that is distinct from \(p\) and \(q\).

We next wish to consider a multiple pendulum type problem, where the potential \(V\) is 1-periodic in all of its arguments. More precisely, we assume:

**PV1** \(V \in C^2(\mathbb{R} \times \mathbb{R}^d, \mathbb{R})\)

**PV2** \(V(t, q)\) is 1-periodic in \(t, q_i\) for \(i = 1, 2, \ldots, d\).

**PV3** \(V(t, \bar{n}) = 0 > V(t, x)\) for all \(\bar{n} \in \mathbb{Z}^d, x \notin \mathbb{Z}^d\).

**PV4** \(V_{qq}(t, \bar{n})\) is negative definite for all \(\bar{n} \in \mathbb{Z}^d, \text{all } t\).

In this case, \(K(V) = \mathbb{Z}^d\), and our results from the previous section do not apply to such a potential. However, we can do the following: If \(v_n\) is a sequence in \(\hat{E}\)
such that $I(v_n)$ is bounded and $v_n(-\infty) = 0$ for all $n$, and $\|v_n\|_{L^\infty} \leq M$ for all $n$. Then we modify the potential outside of $B_{M+1}(0) \subset \mathbb{R}^d$ in such a fashion that the new potential agrees with $V$ on the range of the $v_n$, and yet has only finitely many zeros. As it turns out, Lemma 2.1.3 also applies when $V$ satisfies $PV1 - PV4$. To be more precise, suppose that we have sequence $v_n$ such that

(i) $v_n(-\infty) = 0$ for all $n$

(ii) $I(v_n) < M$

By Lemma 2.1.3, there is a $C(M)$ such that $\|v_n\|_{L^\infty(\mathbb{R})} \leq C(M)$. Let $\tilde{V}(t, x) := V(t, x) - \phi(x)$, where $\phi \in C^\infty$ is chosen such that $\phi(x) \equiv 0$ for $|x| < C(M) + 1$, $\phi(x) \equiv 1$ for $|x| \geq C(M) + 2$, and $0 \leq \phi \leq 1$. Then, we may apply our results to

$$\tilde{I}(v) := \int_{\mathbb{R}} \left( \frac{1}{2} |\dot{v}|^2 - \tilde{V}(t, v) \right) dt.$$ 

We identify $\tilde{I}'(v)$ with $\tilde{J}'(0)$, where $\tilde{J}(u) := \tilde{I}(u + v)$.

Let us recall some important facts about $I$, assuming now that $V$ is 1-periodic in all of its arguments. Proofs may be found in [22].

Let $q_1$ be a minimizer of $I$ in $A_1 := \{v \in \hat{E} \mid v(-\infty) = 0 \text{ and } v(\infty) \neq 0\}$. For $2 \leq i \leq n + 1$, we inductively define

$$A_i := \{v \in \hat{E} \mid v(-\infty) = 0 \text{ and } v(\infty) \neq \sum_{j=1}^{i-1} k_j q_j(\infty) \text{ where } k_j \in \mathbb{Z}, k_j \geq 0\}$$

where $q_j$ is a minimizer of $I$ over $A_j$. More precisely, we define $A_1$, and choose a minimizer $q_1$ of $I$ over $A_1$. Then we define $A_2$ in terms of $q_1$, and find $q_2$ by minimizing $I$ over $A_2$. Then we define $A_3$ and so on.)

Again, if each $q_i$ is an isolated minimizer of $I$ in $A_i$, then

$$c_i := \inf_{h \in \Gamma, 0 \leq s \leq 1} J_i(h_s) > 0,$$
where \( J_i(u) := I(q_i + u) - I(q_i) \) and
\[
\Gamma_i := \{ h \in C([0, 1], E) \mid h_o = 0 \text{ and } h_1(t) = q_i(t + 1) - q_i(t) \}
\]
for \( i = 1, 2, \ldots, n + 1 \). If \( c_i = 0 \), then we may argue as in Proposition 2.2.1 to get a contradiction to the assumption that \( q_i \) is an isolated minimizer of \( I \) in \( A_i \).

Now, we define \( \tilde{V} \) as above with \( M := \max\{c_i\} + 1 \).

**Proposition 2.2.5.** Suppose there is a (PS) sequence \( u_n \) of \( J_i \) such that \( J_i(u_n) \rightarrow b \) for some \( 0 < b < \nu \). Then there is a \( u \) such that \( J_i(u) = b \) and \( J'_i(u) = 0 \).

**Proof.** Let \( v_n := q_i + u_n \). Then we have \( I(v_n) = J_i(u_n) + I(q_i) \rightarrow b + I(q_i) \) and \( I'(v_n) \rightarrow 0 \), so \( v_n \) is a (PS) sequence for \( I \). Now, notice that \( v_n(\mathbb{R}) \subset B_{C(M)}(0) \), so \( v_n \) has support in the region where \( \tilde{V} = V \). Notice that this implies \( \tilde{I}(v_n) = I(v_n) \rightarrow b + I(q_i) \) and \( \tilde{I}'(v_n) = I'(v_n) \rightarrow 0 \), so \( v_n \) is also a (PS) sequence for \( \tilde{I} \).

Then by Theorem 2.1.21, we get a chain of critical points \( \tilde{v}^1, \ldots, \tilde{v}^k \) of \( \tilde{I} \) connecting 0 to \( q_i(\infty) \). Moreover, we have \( b + I(q_i) = \sum_{j=1}^{k} \tilde{I}(\tilde{v}^j) \). We claim that each \( \tilde{v}^j \) is also a critical point of \( I \). This follows from the remark following Theorem 2.1.21:
\[
\|\tilde{v}^j\|_{L^\infty(\mathbb{R})} < C(M) + 1 \text{ for } j = 1, 2, \ldots, k.
\]

Thus, in fact the \( \tilde{v}^j \) are also critical points of \( I \). Notice then that we have \( b + I(q_i) = \sum_{j=1}^{k} \tilde{v}^j \), and \( \tilde{v}^1, \tilde{v}^2, \ldots, \tilde{v}^k \) is a chain connecting 0 and \( q_i(\infty) \). We claim that there is at least one \( \tilde{v}^j \) such that \( \tilde{v}^j - \tilde{v}^j(-\infty) \in A_i \). If not, then we have the following:
\[
\tilde{v}^1 \text{ connects } 0 \text{ to } \sum_{l=1}^{i-1} k_l^1 q_l(\infty) \quad (2.106)
\]

Because of the assumption that there is no \( \tilde{v}^j - \tilde{v}^j(-\infty) \in A_i \), \( \tilde{v}^2 - \tilde{v}^2(-\infty) \notin A_i \).
Hence
\[
\tilde{v}^2(\infty) - \tilde{v}^2(-\infty) = \sum_{l=1}^{i-1} k_l^2 q_l(\infty). \tag{2.107}
\]

But then by (2.106) and (2.107)
\[
\tilde{v}^2(\infty) = \tilde{v}^2(-\infty) + \sum_{l=1}^{i-1} k_l^2 q_l(\infty) = \tilde{v}^1(\infty) + \sum_{l=1}^{i-1} k_l^2 q_l(\infty) = \sum_{l=1}^{i-1} (k_l^1 + k_l^2) q_l(\infty).
\]

Continuing on in this spirit, we get that
\[
\tilde{v}^j(\infty) = \sum_{l=1}^{i-1} \tilde{k}_l q_l(\infty) = q_i(\infty),
\]
which contradicts the definition of $\mathcal{A}_i$. Thus, there is at least one element of the chain that is in $\mathcal{A}_i$. Suppose that it is $\tilde{v}^1$. Notice then that (since $q_i$ is a minimizer of $I$ over $\mathcal{A}_i$) $I(\tilde{v}^1) - I(q_i) \geq 0$, and so
\[
\nu > b \geq I(\tilde{v}^1) - I(q_i) + \sum_{j=2}^{k} I(\tilde{v}^j) \geq (k - 1)\nu.
\]

But then we must have $k = 1$, hence $b = I(\tilde{v}^1) - I(q_i)$. Letting $u := \tilde{v}^1 - q_i$, we have $J_i(u) = I(\tilde{v}^1) - I(q_i) = b$, and since $\tilde{v}^1$ is a critical point of $I$, we have $J_i'(u) = I'(\tilde{v}^1) = 0$. \hfill \Box

**Corollary 2.2.6.** If $c_i < \nu$ for any $i$, then there is a $u \in E$ such that $J_i(u) = c_i$ and $J_i'(u) = 0$.

**Proof.** We can find a (PS) sequence $u_n$ such that $J_i(u_n) \to c_i$ and $J_i'(u_n) \to 0$. Then, we simply apply the preceding proposition. \hfill \Box

Notice that the Corollary implies that we get a new heteroclinic from 0 to $q_i(\infty)$, namely $v := q_i + u$.

By similar arguments, we get an analog of Theorem 2.2.3:
Theorem 2.2.7. Suppose that

\[ c_i \neq \sum_{\text{finite}} I(v^j) \]

where \( v^j \in \{q_1, q_2, \ldots, q_{n+1}\} \cup \{\text{new critical values associated with } c_1, c_2, \ldots, c_{i-1}\} \).

Then, there must be a non-constant \( v \) with \( I(v) < \infty, I'(v) = 0, v \neq q_j \) for any \( j \), and \( v \neq \text{critical points associated with } c_1, c_2, \ldots, c_{i-1} \).

In other words, if \( c_i \) is not a sum of previously known critical values, we must get some new critical point. Again, this new critical point might be a heteroclinic, or a homoclinic. However, if we know that

\[ c_i \neq \sum_{\text{finite}} I(v^j), \]

where

\[ v^j \in \{q_1, q_2, q_{n+1}\} \cup \{\text{new critical values associated with } c_1, c_2, \ldots, c_{i-1}\} \]

\[ \cup \{\text{homoclinics}\}, \]

then we get a new heteroclinic solution. Notice that we do not specify if this new heteroclinic connects 0 to any of \( q_1(\infty), q_2(\infty), \ldots, q_{n+1}(\infty) \). We know only that this new heteroclinic solution \( \tilde{q} \) of (HS) is an element of the chain connecting 0 to \( q_n(\infty) \). Because of the periodicity of \( V \), we can translate \( \tilde{q} \) such that \( \tilde{q}(-\infty) = 0 \), but we do not know if \( \tilde{q}(\infty) \) equals \( q_1(\infty), q_2(\infty), \ldots, q_{n+1}(\infty) \).

How can we verify the assumption that \( c < \nu \)? An admissible \( h \in \Gamma \) is \( h_s(t) := \)
Now, we have

$$J(h_s) = \int_{\mathbb{R}} \frac{1}{2} |\dot{q}(t+s)|^2 - V(t, q(t+s)) \, dt - \int_{\mathbb{R}} \frac{1}{2} |\dot{q}(t)|^2 - V(t, q(t)) \, dt$$

$$= \int_{\mathbb{R}} V(t, q(t)) - V(t, q(t+s)) \, dt$$

changing variables

$$= \int_{\mathbb{R}} V(t, q(t)) - V(t-s, q(t)) \, dt$$

Now, suppose that $V$ has the form $V_\varepsilon(t, x) = (1 + \varepsilon g(t)) f(x)$, where $f, g$ satisfy the following:

(V1) $f$ and $g$ are $C^2$, and $g$ is 1-periodic

(V2) There is a finite set of points $K(f) = \{\xi_1, \xi_2, \ldots, \xi_k\}$ such that $f(x) < 0 = f(\xi_j)$ for all $x \neq \xi_j$.

(V3) $\lim \inf_{|x| \to \infty} f(x) \leq -\alpha < 0$

(V4) $f_{qq}(\xi_j)$ is negative definite for $i = 1, 2, \ldots, k$.

Let $I_\varepsilon(v) := \int_{\mathbb{R}} \left(\frac{1}{2} |v|^2 - V_\varepsilon(t, v)\right) \, dt$. We let $q_\varepsilon$ be a minimizer of $I_\varepsilon$. For simplicity, let us assume that $K(V_\varepsilon) = \{0, \xi\}$ We claim that for $\varepsilon$ suitably small, $c < \nu$. Notice that for $\varepsilon$ suitably small, we have $\frac{1}{2} \leq 1 + \varepsilon g(t) \leq \frac{3}{2}$. This implies that $V_\varepsilon(t, x)$ satisfies (V1)-(V4), because $1 + \varepsilon g(t) \geq \frac{1}{2}$. However, notice that we would like to have $\beta_1, \beta_2$ and $\alpha(r)$ independent of $\varepsilon$. This isn’t necessarily going to happen, but we can bound them from below independently of $\varepsilon$ for $\varepsilon$ small. For example, we have

$$-\frac{1}{2} f(x) \leq -(1 + \varepsilon g(t)) f(x) \leq -\frac{3}{2} f(x),$$
hence $\beta_1$ will be bounded below by the smallest eigenvalue of $-f_{qq}(\xi)$. In much the same fashion, we can bound $\alpha(r)$ away from 0 independently of $\varepsilon$. Now, notice that
\[
c \leq \max_{0 \leq s \leq 1} \int_{\mathbb{R}} V_\varepsilon(t, q_\varepsilon(t)) - V_\varepsilon(t - s, q_\varepsilon(t)) \, dt
\]
\[= \varepsilon \max_{0 \leq s \leq 1} \int_{\mathbb{R}} (g(t) - g(t - s)) f(q_\varepsilon(t)) \]
But $g(t) - g(t - s) = sg'(\zeta)$, hence $|g(t) - g(t - s)| \leq s \|g'\|_{L^\infty} \leq \|g'\|_{L^\infty}$. Thus, we have
\[
c \leq \varepsilon \|g'\|_{L^\infty} \int_{\mathbb{R}} -f(q_\varepsilon(t)) \, dt
\]
If we can show that $\int_{\mathbb{R}} -f(q_\varepsilon) \, dt$ is bounded independent of $\varepsilon$ small, we’ll be done. Recall that we have $-\frac{1}{2}f(x) \leq -(1 + \varepsilon g(t))f(x) \leq -\frac{3}{2}f(x)$ for small enough $\varepsilon$. Thus, we have
\[
\int_{\mathbb{R}} -f(q_\varepsilon) \, dt \leq 2 \int_{\mathbb{R}} -(1 + \varepsilon g(t))f(q_\varepsilon) \, dt
\]
\[\leq 2 \int \frac{1}{2}|q_\varepsilon|^2 - (1 + \varepsilon g(t))f(q_\varepsilon) \, dt
\]
\[= 2I(q_\varepsilon)
\]
\[\leq 2I(\gamma),
\]
where
\[
\gamma(t) := \begin{cases} 
0 & \text{for } t < 0 \\
t\xi & \text{for } 0 \leq t \leq 1 \\
\xi & \text{for } t > 1.
\end{cases}
\]
The last inequality in (2.108) follows from the fact that $q_\varepsilon$ is a minimizer of $I_\varepsilon$ over the class of functions connecting 0 and $\xi$. 
Chapter 3

Gluing Mountain Pass Type Solutions

3.1 Preliminaries

The goal of this section is to prove that there exist infinitely many solutions of (HS) that are close to chains of mountain pass type solutions that we found in the preliminaries. Since this section and the following are very technical and it is hard to see the forest for all of the large trees, we attempt here to provide a map. Roughly speaking, the idea is to take the one-dimensional paths and minimax values that we used to find the original mountain pass type solutions, and put them together to get two-dimensional “paths” and a minimax value. Assuming there are no critical points of the type we seek, a deformation argument is used to get a contradiction to the definition of the minimax value. As is typical with variational gluing, we need some non-degeneracy condition, which is provided by (3.2). Then, we establish some estimates on the size of the derivatives of the functions $\tilde{J}_q$ (see Definition 3.1.1). These estimates are used to prove a deformation type lemma that allows us to deform a given path in $\Gamma_i$ (See Definition 3.1.1) to a path with the property that away from the critical set, the path is bounded away from the
mountain pass critical value. Then, we prove the existence of finite subsets $A_i$ consisting of mountain pass type critical points. Next, we combine these two ideas to produce paths in $\Gamma_i$ that have the property that whenever $\tilde{J}_{n_i}(\gamma_i(\theta_i))$ is close to the mountain pass level $\hat{c}_i$, then $\gamma_i(\theta_i)$ is close to $A_i$.

Ideally, we would like to make our two-dimensional paths to be the sum of the $\gamma_i$, as in [8]. However, this does not make sense, since the mountain pass critical points are really of the form $u + q_1, v + q_2$, where $q_1, q_2$ are minimizers of $I$ over the set of functions in $\hat{E}$ that connecting 0 to $\xi$ ($i = 1$) or connecting $\xi$ to 0 ($i = 2$). Thus, we cannot add paths. Therefore, some approximations need to be defined. Moreover, they have to made appropriately uniform. In this section, a great deal of care is devoted describing the behavior of these approximations. Then, we can make precise what we mean by solutions that shadow a chain of mountain pass type solutions: we will show that there are critical points of $I$ that are close to the $Q((u, v), \rho, k)$ defined in Definition 3.1.18.

Because of the deformation argument we will use, we need bounds away from 0 on $\|I'(x)\|$. Moreover, we need to know that these bounds are sufficiently uniform. Because of the information about how (PS) sequences split, we know that we cannot expect to have uniform bounds everywhere. However, we are able to show that we do have uniform bounds in annular regions about the points $Q((u, v), \rho, k)$ of Definition 3.1.18.

Next, we turn to preparing the minimax value. First, we show that the mountain pass levels of the Preliminaries are independent of the endpoints of the paths used to define them. Then, we define the minimax set $\Gamma(d)$ and minimax value $c(d)$ that we will use. Then, we describe the relationship between the one-dimensional
paths $\gamma_i \in \Gamma_i$ and the elements of the minimax set $\Gamma(d)$, and show an important relationship between the one-dimensional minimax values $\hat{c}_i$ and the minimax value $c(d)$ associated with $\Gamma(d)$.

With these preliminary steps out of the way, we are able to state and prove the main existence result, Theorem 3.2.1. To prove this theorem, we start with “nice” paths in $\Gamma_i$ that have the property that when $\tilde{J}_{q_i}(\gamma_i(\theta_i))$ is close to $\hat{c}_i$, then $\gamma_i(\theta_i)$ is close to $A_i$. Then, using the relationship between these paths and the minimax set, we construct an element $g$ of the minimax set. Assuming that there are no critical points of the sought-after type, we then use a deformation to change $g$ to a $G \in C([0,1], [0,1]^2)$ with the property that $\max_{\theta \in [0,1]^2} I(G(\theta)) < c(d_0)$. However, we no longer know that $G \in \Gamma(d_0)$. Thus, we need to modify $G$ to produce an element of $G \in \Gamma(d_0)$. This modification is done in two steps. In the first step, we replace $G$ on a long interval with minimizers of an appropriate functional, to get a $\bar{G} \in C([0,1], [0,1]^2)$. This step relies a great deal on the fact that the deformation we use does not move $g$ very far. In particular, this knowledge enables us to show the minimizers satisfy (HS). Then, we use the maximum principle to show that on the long interval in question, $\bar{G}(\theta)$ is uniformly close to $\xi$. Because of this, we can change $\bar{G}$ in the middle of these intervals to get an element $G \in \Gamma(d_0)$ such that $\bar{G}$ and $G$ are close enough together that $\max_{\theta \in [0,1]^2} I(G(\theta)) < c(d_0)$, which will give us the contradiction to the definition of the minimax value associated to $\Gamma(d)$.

Throughout, we assume that $V$ satisfies (V1-4), and $K(V) = \{0, \xi\}$. In addition, we will also assume that $V_{qq}$ is bounded, which implies that $I'$ is globally Lipschitz. Throughout, $q_1$ will be a solution of (HS) heteroclinic from 0 to $\xi$ that minimizes $I$ over the set of all elements of $\hat{E}$ connecting 0 to $\xi$, and $q_2$ will be a
solution of (HS) heteroclinic from $\xi$ to 0 that minimizes $I$ over the set of elements of $\hat{E}$ connecting $\xi$ to 0. We make the following definitions:

**Definition 3.1.1.** For any $u \in E$, we define

$$\tilde{J}_{q_i}(u) := J_{q_i}(u) - I(q_i).$$

Then, we define

$$\hat{c}_i := \inf_{h \in \Gamma_i} \max_{s \in [0,1]} \tilde{J}_{q_i}(h(\theta))$$

where

$$\Gamma_i := \{ h \in C([0,1], E) \mid h(0) \equiv 0, \ h(1) \equiv \tau_1 q_i - q_i \}$$

**Remark 3.1.2.** Notice that since $q_i$ is a minimizer of $I$ over the set of $W^{1,2}_{loc}$ functions connecting 0 to $\xi$ or $\xi$ to 0, we must have $\tilde{J}_{q_i}(u) \geq 0$ for all $u \in E$.

We make the following assumption:

$$0 < \hat{c}_i < \nu, \quad (3.1)$$

where $\nu$ is the smallest critical value of $I$ corresponding to a nonconstant solution of (HS). Then, by Theorem 2.1.21, we know there is a $u_i \in E$ such that $J'_{q_i}(u_i) = 0$, and $J_{q_i}(u_i) = \hat{c}_i + I(q_i)$. Thus, $\hat{q}_i := u_i + q_i$ is a solution of (HS) connecting 0 to $\xi$ ($i = 1$) or connecting $\xi$ to 0 ($i = 2$). In order to find solutions of (HS) that shadow the chain $\{\hat{q}_1, \hat{q}_2\}$, we need to make an additional assumption. Before we do that, we fix some notation.
Definition 3.1.3. For any $J \in C^1(E, \mathbb{R})$, we define

\[ K(J) := \{ u \in E \mid J'(u) = 0 \}, \]

\[ (J)^b := \{ u \in E \mid J(u) \leq b \} \]

\[ (J)_a := \{ u \in E \mid J(u) \geq a \} \]

\[ (J)^b_a := (J)_a \cap (J)^b \]

\[ K(J)^b_a := K(J) \cap (J)^b_a \]

Finally, for any $A \subset E$ and any $r > 0$, we put $N_r(A) = \{ u \in E \mid \text{dist}(u, A) \leq r \}$.

By Remark 3.1.2, we must have $(\tilde{J}_q)_a \subset [0, \infty)$ and $(\tilde{J}_q)^b \subset [0, \infty)$. We now assume that there is an $\alpha_1 > 0$ and a $\mu_1 > 0$ such that

\[
\text{if } u \neq w \text{ for } u, w \in K(\tilde{J}_q)^{\hat{c}_i + \alpha_1}_{\hat{c}_i - \alpha_1}, \text{ then } \|u - w\|_E \geq \mu_1. \tag{3.2}
\]

Thus, critical points of $\tilde{J}_q$ in $(\tilde{J}_q)^{\hat{c}_i + \alpha}_{\hat{c}_i - \alpha}$ are uniformly isolated. An example of when (3.2) is satisfied is provided by the following lemma:

Lemma 3.1.4. Suppose that there is an $\alpha > 0$ such that there are only finitely many critical points up to translation by integers in $K(\tilde{J}_q)^{\hat{c}_i + \alpha}_{\hat{c}_i - \alpha}$. Then (3.2) is satisfied.

Proof. If not, then there is a $u$ and a sequence $u_n$ in $K(\tilde{J}_q)^{\hat{c}_i + \alpha}_{\hat{c}_i - \alpha}$ such that

\[ 0 < \|u - u_n\| < \frac{1}{n}. \tag{3.3} \]

Notice that by passing to a subsequence, we may assume that $u_n = \tau_{k_n} v$ for some $v \in K(\tilde{J}_q)^{\hat{c}_i + \alpha}_{\hat{c}_i - \alpha}$ and a sequence $\{k_n\} \in \mathbb{Z}$. In particular, we will have

\[ 0 < \|u - \tau_{k_n} v\| < \frac{1}{n}. \tag{3.4} \]
We claim that in fact, $k_n$ is bounded, and so on a subsequence $u_n \equiv \tau_k v$ for all $n$. But then (3.4) implies that $u = \tau_k v$, which is impossible. Suppose then that $k_n$ is unbounded. Notice that since $v \neq 0$, there is a $t^*$ such that $|v(t^*)| > 0$. Since $u \in E$, $u(t) \to 0$ as $|t| \to \infty$. We know that

$$|u(t^* + k_n) - v(t^*)| = |u(t^* + k_n) - \tau_k v(t^* + k_n)| \leq \|u - \tau_k v\|_{L^\infty}$$

$$\leq C\|u - \tau_k v\| < \frac{C}{n} \to 0 \quad (3.5)$$

as $n \to \infty$. But, if $k_n$ is unbounded, then $u(t^* + k_n) \to 0$ as $n \to \infty$, and so (3.5) implies that $v(t^*) = 0$, which is impossible.

Without loss of generality, we may assume that $\alpha_1$ is sufficiently small that $\hat{c}_i + \alpha_1 < \nu$.

Lemma 3.1.5. There is a $\mu \leq \mu_1$ such that if $u \in (\tilde{J}_q)^{\hat{c}_i+\alpha_1}$ then $B_\mu(u) \subset (\tilde{J}_q)^{\hat{c}_i-\alpha_1}$.

Proof. We have

$$\tilde{J}_{q_i}(v) - \tilde{J}_{q_i}(u) = \int_0^1 \tilde{J}_{q_i}'(sv + (1 - s)u)(v - u) \, ds. \quad (3.6)$$

Since $u \in K(\tilde{J}_q)$ and $\tilde{J}_q'$ is Lipschitz (which follows from the fact that $I'$ is Lipschitz), (3.6) implies

$$|\tilde{J}_{q_i}(v) - \tilde{J}_{q_i}(u)| \leq \int_0^1 \|\tilde{J}_{q_i}'(u + s(v - u))\|\|v - u\| \, ds$$

$$\leq \int_0^1 \|\tilde{J}_{q_i}'(u + s(v - u)) - \tilde{J}_{q_i}'(u)\|\|v - u\| \, ds$$

$$\leq K\|v - u\|^2. \quad (3.7)$$
Thus, if \( \mu := \min\{\mu_1, \sqrt{\frac{\pi}{2K}}\} \), (3.7) implies that if \( v \in B_\mu(u) \), then
\[
|\bar{J}_q(v) - \bar{J}_q(u)| \leq \frac{\alpha_1}{2},
\] (3.8)
and so if \( u \in \mathcal{K}(\bar{J}_q)^{\hat{c}_i + \frac{\alpha_1}{2}} \), then (3.8) implies that
\[
|\bar{J}_q(v) - \hat{c}_i| \leq |\bar{J}_q(v) - \bar{J}_q(u)| + |\bar{J}_q(u) - \hat{c}_i| \leq \frac{\alpha_1}{2} + \frac{\alpha_1}{2},
\]
which finishes the proof.

Corollary 3.1.6. If \( u \in \mathcal{K}(\bar{J}_q)^{\hat{c}_i + \frac{\alpha_1}{2}} \) and \( v \in \mathcal{K}(\bar{J}_q) \) are such that \( \|u - v\| \leq \mu \), then \( u = v \).

Proof. Since \( \|u - v\| \leq \mu \), Lemma 3.1.5 implies that \( v \in (\bar{J}_q)^{\hat{c}_i - \frac{\alpha_1}{2}} \). Then (3.2) implies that \( u = v \) since \( \mu_1 \leq \mu \).

Next, we prove that Corollary 3.1.6 implies that \( \hat{c}_i \) is an isolated critical value of \( \bar{J}_q \).

Lemma 3.1.7. There is an \( \alpha > 0 \) such that for all \( \bar{\alpha} \leq \alpha \) if \( u \in \mathcal{K}(\bar{J}_q)^{\hat{c}_i + \bar{\alpha}} \), then \( \bar{J}_q(u) = \hat{c}_i \).

Proof. Without loss of generality, we may assume that \( \alpha \leq \frac{\alpha_1}{2} \). We consider the case \( i = 1 \), since the proof for \( i = 2 \) is similar. If the statement is false, then there is a sequence \( \{u_n\} \subset E \) such that \( \bar{J}'_{q_1}(u_n) = 0 \) and \( \bar{J}_{q_1}(u_n) \neq \hat{c}_1 \) for all \( n \) and \( \bar{J}_{q_1}(u_n) \to \hat{c}_1 \) as \( n \to \infty \). Let \( p_n := u_n + q_1 \), so \( I'(p_n) = 0 \) for all \( n \) and \( I(p_n) \to \hat{c}_1 + I(q_1) \) as \( n \to \infty \). Thus, by Theorem 1.20 of the Preliminaries, there is a chain \( (v_1, \ldots, v_k) \) of critical points of \( I \) connecting \( 0 \) to \( \xi \). Moreover, we must have
\[
\sum_{j=1}^{k} I(v_j) = \hat{c}_1 + I(q_1). \quad (3.9)
\]
Since \((v_1, \ldots, v_k)\) is a chain connecting 0 and \(\xi\), at least one of the \(v_j\) connects 0 to \(\xi\). For notational convenience, suppose it is \(v_1\). Then, we will have

\[
\nu > \hat{c}_1 = I(v_1) - I(q_1) + \sum_{j=2}^{k} I(v_j) \geq (k - 1)\nu, \tag{3.10}
\]

and so \(k = 1\). But then, by Theorem 1.20 of the preliminaries, this will imply that there are only two intervals on which each \(q_n\) spends an unbounded amount of time close to \(K(V)\). Hence, Proposition 1.12 of the preliminaries implies there is a sequence \(\{k_n\} \subset \mathbb{Z}\) and a critical point \(q \in \hat{E}\) of \(I\) such that

\[
\|\tau_{k_n}p_n - q\|_E \to 0 \tag{3.11}
\]

as \(n \to \infty\) and \(I(q) = \hat{c}_1 + I(q_1)\). Let \(v_n := \tau_{k_n}p_n - q_1\) and let \(v := q - q_1\). Then, (3.11) implies that

\[
\|v_n - v\|_E \to 0 \tag{3.12}
\]

as \(n \to \infty\). Next, notice that since each \(\tau_{k_n}p_n\) is a critical point of \(I\), \(v_n\) is a critical point of \(\tilde{J}_{q_1}\). Moreover, \(\tilde{J}_{q_1}(v_n) = I(\tau_{k_n}p_n) - I(q_1) = I(p_n) - I(q_1) = \tilde{J}_{q_1}(v_n) \neq \hat{c}_1\).

Since \(\tilde{J}_{q_1}(v) = I(q) - I(q_1) = \hat{c}_1\), the \(v_n\) are distinct from \(v\). But then (3.12) contradicts Corollary 3.1.6.

Next, we prove an estimate about the size of \(\tilde{J}_{q_1}'\) away from the set \(\mathcal{K}(\tilde{J}_{q_1})^{\hat{c}_i+\alpha}_{\hat{c}_i-\alpha}\).

**Lemma 3.1.8.** For any \(r > 0\) with \(r < \mu/2\), let

\[
\delta_i(r) := \inf \left\{ \|\tilde{J}_{q_1}'(z)\|_{E'} \mid z \in (\tilde{J}_{q_1})^{\hat{c}_i+\alpha}_{\hat{c}_i-\alpha} \setminus N_r(\mathcal{K}(\tilde{J}_{q_1})^{\hat{c}_i+\alpha}_{\hat{c}_i-\alpha}) \right\}.
\]

Then, \(\delta_i(r) > 0\), and if \(r_1 \leq r_2\), then \(\delta_i(r_1) \leq \delta_i(r_2)\).
Proof. We consider the case when $i = 1$. The proof for $i = 2$ is similar. Suppose that $r < \frac{\alpha}{4}$. If the lemma is false, then there is a sequence $\{z_n\} \subset E$ such that $\tilde{J}_q(z_n) \rightarrow 0$ as $n \rightarrow \infty$, and $\tilde{J}_q(z_n) \in [\hat{c}_1 - \tilde{\alpha}, \hat{c}_1 + \tilde{\alpha}]$, and $\|z_n - v\|_E \geq r$ for all $v \in \mathcal{K}(\tilde{J}_q)_{\hat{c}_1 - \tilde{\alpha}}$. Passing to a subsequence and relabeling, we may assume that $\tilde{J}_q(z_n) \rightarrow a \in [\hat{c}_1 - \tilde{\alpha}, \hat{c}_1 + \tilde{\alpha}]$ as $n \rightarrow \infty$. Thus, $z_n$ is a (PS) sequence for $\tilde{J}_q$.

Let $w_n := q_1 + z_n$. Then, $I(w_n) = \tilde{J}_q(z_n) + I(q_1) \rightarrow a + I(q_1)$ and $I'(w_n) \rightarrow 0$ as $n \rightarrow \infty$. Therefore, by Theorem 2.1.21, there is chain $(v_1, \ldots, v_k)$ connecting $0$ to $\xi$, and $a + I(q_1) = \sum_{i=1}^{k} I(v_i)$. Now, we have $a + I(q_1) \leq \hat{c}_1 + \tilde{\alpha} + I(q_1) < I(q_1) + \nu$. Because $(v_1, \ldots, v_k)$ is a chain connecting $0$ to $\xi$, at least one of the $v_i$'s must connect $0$ to $\xi$. For notational convenience, suppose that $v_1$ connects $0$ to $\xi$. As in (3.10),

$$\nu > \left( \sum_{i=1}^{k} I(v_i) \right) - I(q_1) = (I(v_1) - I(q_1)) + \sum_{i=2}^{k} I(v_i) \geq (k - 1)\nu, \quad (3.13)$$

and so $k = 1$. But then by Theorem 2.1.21, there are exactly two intervals $(-\infty, \tilde{t}_n), (\tilde{t}_n, \infty)$ on which the $w_n$ spend an unbounded (in $n$) amount of time near $K(V)$. Then, by Proposition 2.1.13, there is a sequence $\{k_n\} \subset \mathbb{Z}$ and a $w \in \hat{E}$ such that

$$\|w_n - \tau_{k_n}w\|_E \rightarrow 0 \quad (3.14)$$

as $n \rightarrow \infty$. Moreover, we will have $I'(w) = 0$ and $I(w) = a + I(q_1)$. Notice that by the definition of $w_n$, (3.14) implies that

$$\|z_n - (\tau_{k_n}w - q_1)\|_E \rightarrow 0 \quad (3.15)$$

as $n \rightarrow \infty$. Let $v_n := \tau_{k_n}w - q_1$. If we can show that $v_n \in \mathcal{K}(\tilde{J}_q)_{\hat{c}_1 + \tilde{\alpha}}$, then we will have a contradiction to the assumption that $\|z_n - u\|_E \geq r$ for all $u \in \mathcal{K}(\tilde{J}_q)_{\hat{c}_1 - \tilde{\alpha}}$. 


Notice that we have
\[
\bar{J}_{q_1}(v_n) = I(v_n + q_1) - I(q_1) = I(\tau_{k_n}w) - I(q_1) \\
= I(w) - I(q_1) = a \in [\hat{c}_1 - \hat{\alpha}, \hat{c}_1 + \hat{\alpha}]
\] (3.16)
for all \( n \). Moreover, we have
\[
\|\bar{J}'_{q_1}(v_n)\|_{E'} = \|I'(\tau_{k_n}w)\|_{E'} = \|I'(w)\|_{E'} = 0,
\]
hence \( v_n \in K_{\bar{J}_{q_1}}(\hat{c}_1) \).

Next, we prove that on suitably small neighborhoods of \( K_{\bar{J}_{q_1}}(\hat{c}_1) =: \mathcal{K}(\hat{c}_i) \), \( \bar{J}_{q_1}(u) \) is uniformly close to \( \hat{c}_i \).

**Lemma 3.1.9.** For any \( \varepsilon > 0 \), there is a \( \rho_1 = \rho_1(\varepsilon) \) such that if \( \rho < \rho_1 \) and \( u \in N_\rho(\mathcal{K}(\hat{c}_i)) \), then \( \bar{J}_{q_1}(u) \in [\hat{c}_i - \varepsilon, \hat{c}_i + \varepsilon] \).

**Proof.** Notice if \( u \in N_\rho(\mathcal{K}(\hat{c}_i)) \), then there is a \( v \in \mathcal{K}(\hat{c}_i) \) such that
\[
\bar{J}_{q_1}(u) = \hat{c}_i + \int_0^1 \frac{d}{ds}(\bar{J}_{q_1}(v + s(u - v))) \, ds
\] (3.17)
\[
= \hat{c}_i + \int_0^1 \bar{J}'_{q_1}(v + s(u - v))(u - v) \, ds.
\]
But we also know that \( \bar{J}'_{q_1}(v) = 0 \) and \( \bar{J}'_{q_1} \) is globally Lipschitz, because of the assumption that \( V_{qq} \) is bounded. Thus, for any \( z \in B_\rho(v) \), we have
\[
\|\bar{J}'_{q_1}(z)\|_{E'} = \|\bar{J}'_{q_1}(z) - \bar{J}'_{q_1}(v)\|_{E'} \leq K\|z - v\| \leq K\rho
\] (3.18)
Combining (3.17) and (3.18), we see that
\[
|\bar{J}_{q_1}(u) - \hat{c}_i| \leq \int_0^1 \|\bar{J}'_{q_1}(v + s(u - v))\|_{E'} ds\|u - v\|
\leq K\rho^2.
\] (3.19)
Thus, if we take \( \rho_1 := \sqrt{\varepsilon/K} \), the lemma holds. \( \square \)
Next, we want to prove a deformation result which we will use to manufacture a homotopy \( h_i \in \Gamma_i \) from 0 to \( \tau_1 q_i - q_i \) with the property that when \( \tilde{J}_{q_i}(h_i(s)) \) is close to \( \hat{c}_i \), then \( h_i(s) \) is close to \( K(\tilde{J}_{q_i})_{\hat{c}_i - \alpha} \).

**Proposition 3.1.10.** For any \( \varepsilon \in (0, \hat{\alpha}] \) and \( r < \mu/2 \), there is an \( \varepsilon \in (0, \varepsilon) \), a \( \zeta_i \in C([0, 1] \times E, E) \) and a \( \sigma_i \in C((\tilde{J}_{q_i})_{\hat{c}_i + \varepsilon}, [0, 1]) \) such that

1. \( \zeta_i(0, x) = x \) for all \( x \in E \).
2. \( \zeta_i(s, x) = x \) if \( x \notin (\tilde{J}_{q_i})_{\hat{c}_i - \varepsilon} \)
3. \( \tilde{J}_{q_i}(s, x) \) is non-increasing in \( s \)
4. \( \zeta_i \left( 1, (\tilde{J}_{q_i})_{\hat{c}_i + \varepsilon} \right) \setminus N_r \left( K(\tilde{J}_{q_i})_{\hat{c}_i - \varepsilon} \right) \subset (\tilde{J}_{q_i})_{\hat{c}_i - \varepsilon} \)
5. \( \sigma_i(x) = 0 \) if \( x \notin (\tilde{J}_{q_i})_{\hat{c}_i - \varepsilon} \setminus N_r \left( K(\tilde{J}_{q_i})_{\hat{c}_i - \varepsilon} \right) \), and \( \tilde{J}_{q_i}(\zeta_i(x), x) = \hat{c}_i - \varepsilon \) for all \( x \in (\tilde{J}_{q_i})_{\hat{c}_i + \varepsilon} \setminus N_r \left( K(\tilde{J}_{q_i})_{\hat{c}_i - \varepsilon} \right) \)
6. \( \|\zeta_i(\sigma_i(x), x) - x\|_E \leq r \) for all \( x \in (\tilde{J}_{q_i})_{\hat{c}_i + \varepsilon} \).

**Proof.** We consider only the case \( i = 1 \), and drop the subscript \( i \) for notational convenience. Let

\[
\varepsilon := \min \left\{ \frac{1}{2} \hat{\varepsilon}, \frac{r\delta(r/8)}{32} \right\} \tag{3.20}
\]

where \( \delta(r/8) \) is the bound from below on \( \tilde{J}_{q_i}(x) \) for \( x \in (\tilde{J}_{q_i})_{\hat{c}_i + \hat{\alpha}} \setminus N_{r/8} \left( K(\tilde{J}_{q_i})_{\hat{c}_i - \hat{\alpha}} \right) \) provided by Lemma 3.1.8. Let

\[
f(x) := \frac{\text{dist} \left( x, (\tilde{J}_{q_i})_{\hat{c}_i - \varepsilon} \cup (\tilde{J}_{q_i})_{\hat{c}_i + \varepsilon} \right)}{\text{dist} \left( x, (\tilde{J}_{q_i})_{\hat{c}_i - \varepsilon} \right) + \text{dist} \left( x, (\tilde{J}_{q_i})_{\hat{c}_i + \varepsilon} \right)}. \tag{3.21}
\]
Next, notice that since $\bar{\varepsilon} < \bar{\varepsilon}$, we must have $K(\bar{J}_q)^{\hat{c} + \hat{\alpha}} = K(\hat{c})$. By Lemma 3.1.9, there is a $\rho(\varepsilon) > 0$ such that

$$
\text{if } u \in N_\rho(K(\hat{c})) \text{, then } \bar{J}_q(u) \in [\hat{c} - \varepsilon/2, \hat{c} + \varepsilon/2].
$$

(3.22)

Without loss of generality, we may assume that $\rho < r$. Now, let

$$
\varphi(x) := \frac{\text{dist}(x, N_\rho/16(K(\hat{c})))}{\text{dist}(x, N_\rho/16(K(\hat{c}))) + \text{dist}(x, E \setminus N_{r/8}(K(\hat{c})))}.
$$

(3.23)

Let $D : E' \to E$ be the duality map, and define $\nabla \bar{J}_q(x) := D\bar{J}_q'(x)$. Next, we define

$$
W(x) := \begin{cases} 
\bar{\varepsilon}\varphi(x)f(x)\frac{\nabla \bar{J}_q(x)}{\|\bar{J}_q'(x)\|_E} & \text{if } \bar{J}_q'(x) \neq 0 \\
0 & \text{if } \bar{J}_q'(x) = 0
\end{cases}
$$

(3.24)

We claim that $W(x)$ is locally Lipschitz. To verify our claim, note that since $\varphi$, $f$ and $\bar{J}_q'$ are locally Lipschitz, it suffices to show that for any $x$ with $\bar{J}_q'(x) = 0$, there is an $r^* = r^*(x) > 0$ such that $f(y)\varphi(y) = 0$ for all $y \in B_{r^*/8}(x)$. If there is no such $r^*$, then there is a sequence $y_n$ such that $f(y_n)\varphi(y_n) \neq 0$ for all $n$ and $\|y_n - x\| \to 0$ as $n \to \infty$. Now, since $f(y_n) \neq 0$, we must have $\bar{J}_q(y_n) \in [\hat{c} - \bar{\varepsilon}, \hat{c} + \bar{\varepsilon}]$ for all $n$. Since $\bar{J}_q(y_n) \to \bar{J}_q(x)$ as $n \to \infty$, Lemma 3.1.7 implies that $\bar{J}_q(x) = \hat{c}$. Therefore, $y_n$ must get close to $K(\hat{c})$. However, since $\varphi(y_n) \neq 0$, $\|y_n - u\|_E \geq \rho/16$ for all $u \in K(\hat{c})$, which is impossible.

Since $W$ is locally Lipschitz, there is a unique solution $\zeta(s, x)$ of

$$
\frac{d\zeta}{ds} = -W(\zeta) \quad \zeta(0, x) = x.
$$

(3.25)

We now claim that $W(x)$ is bounded, and so the solution $\zeta$ exists for all $s$. To verify our claim, notice that if $W(x) \neq 0$, then we must in particular have $\varphi(x)f(x) \neq 0$. 

Thus, \( x \in (\tilde{J}_q)^{\hat{c} + \hat{\alpha}} N_{\rho/16}(\mathcal{K}(\hat{c})) \), and so by Lemma 3.1.8, we must have \( \|\tilde{J}'_q(x)\|_{E'} \geq \delta(\frac{\rho}{16}) \). Thus, we have

\[
\|W(x)\|_E \leq \left| f(x)\varphi(x) \right| \frac{\varepsilon}{\|J'_q(x)\|_{E'}} \leq \frac{\varepsilon}{\delta(\frac{\rho}{16})} 
\]  

(3.26)

Now, because of the definition of \( \zeta \), (1) is satisfied. Moreover, since \( f(x) = 0 \) for \( x \not\in (\tilde{J}_q)^{\hat{c} + \varepsilon} \), (2) is satisfied. Next, to see that (3) holds, notice that if \( \tilde{J}'_q(x) \neq 0 \), then

\[
\frac{d}{ds} \tilde{J}_q(\zeta(s, x)) = \tilde{J}'_q(\zeta(s, x))W(\zeta(s, x)) = -\varepsilon f(\zeta(s, x))\varphi(\zeta(s, x)) \frac{\tilde{J}'_q(\zeta(s, x))\nabla \tilde{J}_q(\zeta(s, x))}{\|\tilde{J}'_q(\zeta(s, x))\|^2} 
\]  

(3.27)

\[
= -\varepsilon f(\zeta(s, x))\varphi(\zeta(s, x)) \leq 0.
\]

If \( \tilde{J}'_q(x) = 0 \), then \( \zeta(s, x) \equiv x \), and so (3) holds.

To prove (4), we show the existence of a \( \sigma \) satisfying (5). Notice that (5) and (3) imply (4). Suppose that \( x \in (\tilde{J}_q)^{\hat{c} + \varepsilon} N_{\rho/4}(\mathcal{K}(\hat{c})) \). We first show that there is an \( s > 0 \) (depending on \( x \)) such that either \( \zeta(s, x) \in \partial N_{\rho/8}(\mathcal{K}(\hat{c})) \) or \( \zeta(s, x) \in \partial(\tilde{J}_q)^{\hat{c} - \varepsilon} \).

If this is false, then \( f(\zeta(s, x))\varphi(\zeta(s, x)) = 1 \) for all \( s > 0 \). Thus, by (3.27),

\[
\tilde{J}_q(\zeta(s, x)) = \tilde{J}_q(x) + \int_0^s \frac{d}{ds} \tilde{J}_q(\zeta(s, x))ds 
\]  

(3.28)

\[
= \tilde{J}_q(x) - s\varepsilon \to -\infty
\]

as \( s \to \infty \), which contradicts the assumption that \( \zeta(s, x) \) does not reach \( \partial(\tilde{J}_q)^{\hat{c} - \varepsilon} \).

Next, we claim that with \( \varepsilon \) defined as in (3.20), if \( x \in (\tilde{J}_q)^{\hat{c} + \varepsilon} N_{\rho/4}(\mathcal{K}(\hat{c})) \), then \( \zeta(s, x) \) reaches \( \partial(\tilde{J}_q)^{\hat{c} - \varepsilon} \) before it reaches \( \partial N_{\rho/8}(\mathcal{K}(\hat{c})) \). If this is false, there is an interval of the form \([0, \theta(x)]\) such that for all \( s \in [0, \theta(x)] \), one has

\[
\tilde{J}_q(\zeta(s, x)) \in [\hat{c} - \varepsilon, \hat{c} + \varepsilon] \text{ and } dist(\zeta(s, x), \mathcal{K}(\hat{c})) \geq \frac{r}{8}.
\]  

(3.29)
Thus, \( f(\zeta(s, x)) \varphi(\zeta(s, x)) = 1 \) for all \( s \in [0, \theta(x)] \). Since \( \tilde{J}_q(\zeta(s, x)) \geq \hat{c} - \varepsilon \), we have

\[
2\varepsilon \geq \tilde{J}_q(x) - \tilde{J}_q(\zeta(\theta(x), x)) = \int_{\theta(x)}^{0} \frac{d}{ds} \tilde{J}_q(\zeta(s, x)) ds \\
= -\varepsilon \int_{\theta(x)}^{0} ds \\
= \theta(x)\varepsilon.
\]

(3.30)

On the other hand, since \( \zeta(s, x) \) travels from outside \( N_{r/4}(\mathcal{K}(\hat{c})) \) to \( \partial N_{r/8}(\mathcal{K}(\hat{c})) \), we must have

\[
\frac{r}{8} \leq \|\zeta(\theta(x), x) - x\|_E \leq \left\| \int_{0}^{\theta(x)} \frac{d}{ds} \zeta(s, x) ds \right\|_E \\
\leq \int_{0}^{\theta(x)} \|W(\zeta(s, x))\|_E ds \\
\leq \varepsilon \left( \frac{1}{\delta(\frac{r}{8})} \right) \theta(x)
\]

(3.31)

since for \( s \in [0, \theta(x)] \), we have \( \zeta(s, x) \notin \overline{N_{r/8}(\mathcal{K}(\hat{c}))} \) and thus by Lemma 3.1.8, we will have \( \|\tilde{J}_q'(z)\|_E \geq \delta(r/8) \). Combining (3.30) and (3.31), we must have

\[
\frac{r\delta(\frac{r}{8})}{8} \leq \varepsilon \theta(x) \leq 2\varepsilon,
\]

(3.32)

so

\[
\frac{r\delta(\frac{r}{8})}{16} \leq \varepsilon.
\]

(3.33)

But, by (3.20) and (3.33), we have

\[
\varepsilon \leq \frac{1}{2} \left( \frac{r\delta(\frac{r}{8})}{16} \right) < \varepsilon,
\]

(3.34)

which is impossible. Thus, \( \zeta(s, x) \) reaches \( \partial(\tilde{J}_q)^{\hat{c} - \varepsilon} \) before it reaches \( \partial N_{r/8}(\mathcal{K}(\hat{c})) \).
How far is $\zeta(\theta(x), x)$ from $x$, if $\theta(x)$ is the time when $\zeta(s, x)$ reaches $\partial J^\circ_{q-\varepsilon}$?

$$
\|\zeta(\theta(x), x) - x\| = \left\| \int_0^{\theta(x)} \frac{d}{ds} \zeta(s, x) ds \right\|
$$

$$
\leq \int_0^{\theta(x)} \| W(\zeta(s, x)) \| ds
$$

$$
\leq \int_0^{\theta(x)} \| f(\zeta(s, x)) \varphi(\zeta(s, x)) \| \frac{\| \nabla J_q(\zeta(s, x)) \|}{\| J_q(\zeta(s, x)) \|^2} ds
$$

$$
\leq \frac{\varepsilon \theta(x)}{\delta \left( \frac{r}{8} \right)}.
$$

But, by (3.30), we also have $2\varepsilon \geq \varepsilon \theta(x)$. Combining this inequality with (3.35) and (3.20), we see that

$$
\|\zeta(\theta(x), x) - x\| \leq \frac{\varepsilon \theta(x)}{\delta \left( \frac{r}{8} \right)} \leq 2\varepsilon \frac{\delta \left( \frac{r}{8} \right)}{\delta \left( \frac{r}{8} \right)} \leq 2 r \frac{\delta \left( \frac{r}{8} \right)}{16} = r.
$$

In addition, since $2\varepsilon \geq \varepsilon \theta(x)$, we must have

$$
\theta(x) \leq 2 \frac{\varepsilon}{\varepsilon} \leq 1
$$

by (3.20). Thus, (6) holds for $x \in (\tilde{J}_q)_{\tilde{c}+\varepsilon} \cap N_{r/4}(K(\tilde{c}))$ (assuming for now that $\theta(x)$ is continuous in $x$).

Next, suppose that $x \in (\tilde{J}_q)_{\tilde{c}+\varepsilon} \cap N_{r/4}(K(\tilde{c}))$. Thus, we must have $x \in B_{r/4}(u)$ for some $u \in K(\tilde{c})$. We have two possibilities: either (a) $\zeta(s, x)$ remains in $B_{r/2}(u)$ for all $s \in [0, 1]$ or (b) $\zeta(s, x)$ leaves $B_{r/2}(u)$ at some $s \in (0, 1]$. We claim that if (b) occurs, then $\zeta(s, x)$ reaches $\partial(\tilde{J}_q)^{\tilde{c}-\varepsilon}$ before it reaches $\partial B_{r/2}(u)$. Suppose this is false. Then, there must be an interval $(s_1, s_2)$ such that if $s \in (s_1, s_2)$, then $\zeta(s, x) \in (\tilde{J}_q)_{\tilde{c}+\varepsilon} \cap (B_{r/2}(u) \setminus B_{r/4}(u))$ and moreover $\zeta(s_1, x) \in \partial B_{r/4}(u)$, $\zeta(s_2, x) \in \partial B_{r/2}(u)$. Notice that this implies that $f(\zeta(s, x)) \varphi(\zeta(s, x)) = 1$ for all $s \in (s_1, s_2)$. 
We must have
\[ 2\varepsilon \geq \bar{J}_q(\zeta(s_1, x)) - \bar{J}_q(\zeta(s_2, x)) = \int_{s_2}^{s_1} \frac{d}{ds} \bar{J}_q(\zeta(s, x)) ds = \int_{s_1}^{s_2} \bar{\varepsilon} ds = \bar{\varepsilon}(s_2 - s_1), \]
and
\[ \frac{r}{4} \leq \|\zeta(s_2, x) - \zeta(s_1, x)\| \leq \int_{s_1}^{s_2} \left\| \frac{d}{ds} \bar{J}_q(\zeta(s, x)) \right\| ds \leq \frac{\bar{\varepsilon}(s_2 - s_1)}{\delta\left(\frac{\varepsilon}{\delta}\right)}. \]
Combining (3.38) and (3.39), we see that
\[ \frac{r\delta\left(\frac{\varepsilon}{\delta}\right)}{4} \leq 2\varepsilon \leq \frac{r\delta\left(\frac{\varepsilon}{\delta}\right)}{16}, \]
which is impossible. Thus, if (b) occurs, then there must be a \( \theta(x) \in (0, 1] \) such that \( \bar{J}_q(\zeta(s, x)) = \hat{c} - \varepsilon \) and \( \zeta(s, x) \in B_{r/2}(u) \). Notice that this implies that \( \|\zeta(\theta(x), x) - x\| \leq r \). Finally, notice that if (a) occurs, then \( \|\zeta(1, x) - x\| \leq r \).

To finish the proof, we need to show that \( \theta(x) \) is continuous and how to get a \( \sigma \in C((\bar{J}_q) \hat{c} + \varepsilon, [0, 1]) \). For any \( x \in (\bar{J}_q) \hat{c} + \varepsilon \), we define
\[ \omega(x) := \inf\{s \geq 0 \mid \bar{J}_q(\zeta(s, x)) \leq \hat{c} - \varepsilon\} \]
We now show that \( \omega \) is a continuous function from \( (\bar{J}_q) \hat{c} + \varepsilon \) into \([0, \infty]\) in the sense that if \( x_n \to x \), then \( \omega(x_n) \to \omega(x) \) as \( n \to \infty \). Note that we allow the possibility that \( \omega(x) = \infty \), in which case we show that \( \omega(x_n) \to \infty \) as \( n \to \infty \). We consider three cases: (1*) \( \omega(x) = \infty \), (2*) \( \omega(x) \in (0, \infty) \) and finally (3*) \( \omega(x) = 0 \).

In case (1*), let \( x_n \) be a sequence with \( x_n \to x \) as \( n \to \infty \). We need to show that \( \omega(x_n) \to \infty \) as \( n \to \infty \). If this is false, then passing to a subsequence, there
is an $M$ such that $\omega(x_n) \leq M$ for all $n$. But then $\tilde{J}_q(\zeta(M, x_n)) \leq c - \varepsilon$ for all $n$. Since $x \mapsto \zeta(M, x)$ is continuous, we must have $\tilde{J}_q(\zeta(M, x)) \leq c - \varepsilon$, and so $\omega(x) \leq M$, which is impossible. Thus, we must have $\omega(x_n) \to \infty$ as $n \to \infty$.

In case (2$^\ast$), suppose that $\omega(x) = h$. The argument of case (1$^\ast$) above shows that if $x_n \to x$ as $n \to \infty$, then we cannot have $\omega(x_n) \leq \omega(x) - \eta$ for any $\eta > 0$ and all large $n$. Thus, we must have $\omega(x) \leq \liminf \omega(x_n)$. Suppose that there is an $\eta > 0$ such that $\omega(x_n) \geq \omega(x) + \eta$ for all $n$. We claim now that $\tilde{J}_q(\zeta(s, x))$ is strictly decreasing. Since $\omega(x) \in (0, \infty)$, we cannot have $W(x) = 0$, since if $W(x) = 0$, then $\zeta(s, x) \equiv x$ and so $\tilde{J}_q(\zeta(s, x)) = \tilde{J}_q(x)$. But since $\omega(x) > 0$, we must have $\tilde{J}_q(x) > c - \varepsilon$, and so $\omega(x) = \infty$ (since $\zeta(s, x)$ does not move $x$), which is impossible. Thus, (since $W(x) \neq 0$ implies $W(\zeta(s, x)) \neq 0$ for all $s > 0$) we must have $\frac{d}{ds} \tilde{J}_q(\zeta(s, x)) = -\varepsilon \varphi(\zeta(s, x)) f(\zeta(s, x)) < 0$ for all $s$, which proves our claim. Notice that because $\tilde{J}_q(\zeta(s, x))$ is strictly decreasing, we must have $\tilde{J}_q(\zeta(\omega(x) + \eta/2, x)) < c - \varepsilon$. But then $\tilde{J}_q(\zeta(\omega(x) + \eta/2, x_n)) < c - \varepsilon$ for all sufficiently large $n$. Hence for all large $n$

$$\omega(x) + \eta \leq \omega(x_n) \leq \omega(x) + \eta/2,$$  \hspace{1cm} (3.42)

which is impossible.

Finally, we consider case (3$^\ast$). We have two possibilities: either $\tilde{J}_q(x) < c - \varepsilon$, or $\tilde{J}_q(x) = c - \varepsilon$. If $\tilde{J}_q(x) < c - \varepsilon$, then for any sequence $x_n$ with $x_n \to x$ as $n \to \infty$, we will eventually have $\tilde{J}_q(x_n) < c - \varepsilon$, hence $\omega(x_n) = 0$. Suppose now that $\tilde{J}_q(x) = c - \varepsilon$. We claim that $W(x) \neq 0$. Accepting this for the moment, the argument of case (2$^\ast$) will imply that $\tilde{J}_q(\zeta(s, x))$ is strictly decreasing and hence
by (3.42) we will have continuity. To verify our claim, notice that \( f(x) = 1 \) for any \( x \) with \( \bar{J}_q(x) = \hat{c} - \varepsilon \). Next, because of the choice of \( \rho \) in (3.22), we have \( N_{\rho/8}(K(\hat{c})) \subset (\bar{J}_q)_{\hat{c} - \varepsilon /2}^{\hat{c}+\varepsilon /2} \). Thus, we must have \( \varphi(x) \neq 0 \). Finally, because \( \varepsilon < \tilde{\alpha} \), we cannot have \( x \in K(\bar{J}_q) \). Thus, \( W(x) \neq 0 \).

Finally, we may define \( \sigma(x) \) by

\[
\sigma(x) := \min\{1, \omega(x)\} \quad (3.43)
\]

Then, \( \sigma \in C((\bar{J}_q)^{\hat{c}+\varepsilon}, [0, 1]) \). Notice that for \( x \in (\bar{J}_q)^{\hat{c}+\varepsilon} \setminus N_{\varepsilon/4}(K(\hat{c})) \), because \( \theta(x) \leq 1 \), we must have \( \theta(x) = \sigma(x) \), and so (5) and (6) are satisfied. Similarly, if \( x \in (\bar{J}_q)^{\hat{c}+\varepsilon} \cap N_{\varepsilon/4}(K(\hat{c})) \), we will have (6) satisfied for both cases (a) and (b), as shown by the arguments from (3.38) through (3.40).

Next, we prove the existence of some “special” finite subsets of \( K(\hat{c}) \).

**Proposition 3.1.11.** For \( i = 1, 2 \), there exists a finite set \( A_i \subset K(\hat{c}_i) \) such that for any \( \bar{\varepsilon} \leq \tilde{\alpha}/2 \) and any \( \bar{r} \leq \mu/64 \), there is an \( \varepsilon' \in (0, \bar{\varepsilon}) \) and a \( g_i \in \Gamma_i \) such that

\[
(1) \quad \max_{\theta \in [0, 1]} \bar{J}_{\bar{\varepsilon}}(g_i(\theta)) \leq \hat{c}_i + \varepsilon'/4 \\
(2) \quad \text{if } \bar{J}_{\bar{\varepsilon}}(g_i(\theta)) > \hat{c}_i - \varepsilon', \text{ then } g_i(\theta) \in N_{\bar{r}}(A_i).
\]

**Proof.** To simplify the notation, we discard the subscript \( i \). First, we describe the set \( A \). For this, we consider the case when \( \bar{\varepsilon} = \tilde{\alpha}/2 \) and \( \bar{r} = \mu/64 \). We invoke Proposition 3.1.10 with \( \bar{\varepsilon} = \tilde{\alpha}/2 \) and \( \bar{r} = \mu/64 \) to get

\[
\varepsilon_1 = \min\{\tilde{\alpha}/4, \frac{\mu(\varepsilon_{\gamma, 3})}{2 \cdot 64 \cdot 32}\} = \min\left\{\frac{\tilde{\alpha}}{4}, \frac{\mu(\varepsilon_{\gamma, 3})}{2 \cdot 64 \cdot 32}\right\},
\]

a \( \zeta_1 \in C([0, 1] \times E, E) \) and a \( \sigma_1 \in C((\bar{J}_q)^{\hat{c}+\varepsilon_1}, [0, 1]) \). Pick \( g_0 \in \Gamma \) such that

\[
\max_{\theta \in [0, 1]} \bar{J}_q(g_0(\theta)) \leq \hat{c} + \frac{\varepsilon_1}{4}, \quad (3.44)
\]
and define

$$g_1(\theta) := \zeta_1(\sigma_1(g_0(\theta)), g_0(\theta))$$  \hfill (3.45)

Notice that by (3) of Proposition 3.1.10, we automatically have

$$\max_{\theta \in [0,1]} \tilde{J}_q(g_1(\theta)) \leq \hat{c} + \frac{\varepsilon_1}{4}.$$  \hfill (3.46)

Moreover, since $\hat{c} - \tilde{\alpha} > 0$ and $\tilde{J}_q(g_0(0)) = 0 = \tilde{J}_q(g_0(1))$, (2) of Proposition 3.1.10 implies that $g_1 \in \Gamma$. Suppose now that $\tilde{J}_q(g_1(\theta)) > \hat{c} - \varepsilon_1$. By (5) of Proposition 3.1.10, we must have $g_0(\theta) \in (\tilde{J}_q)^{\hat{c}+\varepsilon_1} \cap N_r(\mathcal{K}(\hat{c}))$. But then by (6) of Proposition 3.1.10, we must have $g_1(\theta) \in N_{2r}(\mathcal{K}(\hat{c})) = N_r(\mathcal{K}(\hat{c}))$. We claim that in fact there is a finite subset $A$ of $\mathcal{K}(\hat{c})$ such that if $\tilde{J}_q(g_1(\theta)) > \hat{c} - \varepsilon_1$, then $g_1(\theta) \in N_r(A)$. Suppose that this is not the case. Then, there must exist infinitely many distinct $u_n \in \mathcal{K}(\hat{c})$ and infinitely many distinct $\theta_n \in [0,1]$ such that

$$\|u_n - g_1(\theta_n)\| \leq \tilde{r} = \frac{\mu}{64}$$  \hfill (3.47)

Passing to a subsequence, we may assume that $\theta_n \to \bar{\theta}$ in $[0,1]$ as $n \to \infty$. Thus, $g_1(\theta_n) \to g_1(\bar{\theta})$ as $n \to \infty$ for some $\bar{\theta} \in [0,1]$. But, then for all suitably large $n$, we will have

$$\|u_n - g_1(\bar{\theta})\| \leq \|u_n - g_1(\theta_n)\| + \|g_1(\theta_n) - g_1(\bar{\theta})\| \leq \frac{\mu}{32},$$  \hfill (3.48)

and so for all large $n$

$$\|u_n - u\| \leq \frac{\mu}{16}$$  \hfill (3.49)

which contradicts Corollary 3.1.6. Notice that we have proved the theorem for the particular choice of $\varepsilon = \frac{\tilde{\alpha}}{2}$ and $\tilde{r} = \frac{\mu}{64}$. 
Next, suppose that \( \bar{\varepsilon} < \frac{\tilde{\alpha}}{2} \) and \( \bar{r} < \frac{\mu}{64} \). We again invoke Proposition 3.1.10, with this choice of \( \bar{\varepsilon} \) and \( r := \frac{\bar{r}}{2} \). Let \( \varepsilon_2 = \min\{\frac{\bar{\varepsilon}}{2}, \frac{\bar{r}}{32}\} = \min\{\frac{\bar{\varepsilon}}{2}, \frac{\bar{r}}{64}\} \), \( \zeta_2 \in C([0,1] \times E, E) \) and \( \sigma_2 \in C((\tilde{J}_q)^{\tilde{c}+\varepsilon_2}, [0,1]) \) be those of Proposition 3.1.10. Notice that \( \varepsilon_2 \leq \varepsilon_1 \), since \( \bar{\varepsilon} < \frac{\tilde{\alpha}}{2}, \bar{r} < \frac{\mu}{64} \) and \( \delta \) is decreasing in \( r \) by Lemma 3.1.8.

Ideally, we would like to take \( g(\theta) = \zeta_2(\sigma_2(g_1(\theta)), g_1(\theta)) \), but we do not know that \( g_1([0,1]) \subset (\tilde{J}_q)^{\tilde{c}+\varepsilon_2} \), and so this reasonable definition of \( g_2 \) might not work. We first need to form an intermediate path \( g_{3/2} \in \Gamma \) such that \( g_{3/2}([0,1]) \subset (\tilde{J}_q)^{\tilde{c}+\frac{\varepsilon_2}{2}} \).

By Lemma 3.1.9, we may choose \( \rho_1 < \frac{\mu}{64} \) sufficiently small that

\[
\max_{x \in N_{\rho_1}(\mathcal{K}(\hat{c}))} \tilde{J}_q(x) < \hat{c} + \frac{\varepsilon_2}{4}.
\]

Then, we define

\[
\hat{\phi}(x) := \frac{\text{dist}(x, N_{\rho_1/8}(\mathcal{K}(\hat{c})))}{\text{dist}(x, N_{\rho_1/8}(\mathcal{K}(\hat{c}))) + \text{dist}(x, E \setminus N_{\rho_1/4}(\mathcal{K}(\hat{c})))}
\]

and

\[
\hat{f}(x) := \frac{\text{dist}(x, (\tilde{J}_q)^{\hat{c} - \frac{\alpha}{2}} \cup (\tilde{J}_q)^{\hat{c} + \frac{\alpha}{2}})}{\text{dist}(x, (\tilde{J}_q)^{\hat{c} - \frac{\alpha}{2}} \cup (\tilde{J}_q)^{\hat{c} + \frac{\alpha}{2}}) + \text{dist}(x, (\tilde{J}_q)^{\hat{c} + \frac{\alpha}{4}})}.
\]

Next, define \( W_{3/2}(x) \) by

\[
W_{3/2}(x) := \begin{cases} 
\frac{\varepsilon_1}{4} \hat{\phi}(x) \hat{f}(x) \frac{\nabla \tilde{J}_q(x)}{\|\nabla \tilde{J}_q(x)\|_{L^r}} & \text{if } \tilde{J}_q(x) \neq 0 \\
0 & \text{if } \tilde{J}_q(x) = 0
\end{cases}
\]

and suppose that \( \psi \) satisfies

\[
\frac{d}{ds} \psi = -W_{3/2}(\psi), \quad \psi(0, x) = x.
\]

As in the proof of Proposition 3.1.10, the solution \( \psi(s, x) \) exists for all \( s \in \mathbb{R} \), since \( W_{3/2}(x) \) is bounded. Now, we define our intermediate \( g_{3/2} \) by

\[
g_{3/2}(\theta) := \psi(1, g_1(\theta)).
\]
We claim that

$$\max_{\theta \in [0,1]} \tilde{J}_q(g_{3/2}(\theta)) \leq \hat{c} + \frac{\varepsilon_2}{4}. \tag{3.56}$$

If not, then there is a $\tilde{\theta} \in [0,1]$ such that $\tilde{J}_q(\psi(s, g_1(\tilde{\theta}))) > \hat{c} + \frac{\varepsilon_2}{4}$ for all $s \in [0,1]$. But then $\tilde{c} + \frac{\varepsilon_1}{4} < \tilde{J}_q(\psi(s, g_1(\tilde{\theta}))) < \tilde{c} + \frac{\varepsilon_1}{4}$ for all $s \in [0,1]$. In addition, because of the choice of $\rho_1$ in (3.50), we must also have $\psi(s, g_1(\tilde{\theta})) \not\in N_{\rho_1}(K(\hat{c}))$ for all $s \in [0,1]$. Thus, we will have $\hat{\phi}(\psi(s, g_1(\tilde{\theta}))) \hat{f}(\psi(s, g_1(\tilde{\theta}))) = 1$ for all $s \in [0,1]$. But then $\varepsilon_1 - \varepsilon_2 \geq \tilde{J}_q(g_1(\tilde{\theta})) - \tilde{J}_q(\psi(1, g_1(\tilde{\theta}))) = \int_0^1 \frac{d}{ds} \tilde{J}_q(\psi(s, g_1(\tilde{\theta}))) ds = \int_0^1 \varepsilon_1 \frac{ds}{4} = \frac{\varepsilon_1}{4},$

which is impossible. Thus, (3.56) holds. Finally, we define

$$g(\theta) := \zeta_2(\sigma_2(g_{3/2}(\theta)), g_{3/2}(\theta)). \tag{3.57}$$

Notice that by (3) of Proposition 3.1.10, we must clearly have (1) of Proposition 3.1.11 satisfied, with $\varepsilon' := \varepsilon_2$. Suppose now that $\tilde{J}_q(g(\theta)) > \hat{c} - \varepsilon_2$. By (5) of Propositon 3.1.10, we must have $g_{3/2}(\theta) \in N_r(K(\hat{c}))$, and so by (6) of Proposition 3.1.10, we must have $g(\theta) \in N_{2r}(K(\hat{c})) = N_r(K(\hat{c}))$. However, we want $g(\theta) \in N_r(A)$. Suppose that this is not the case, i.e. $g(\theta) \in N_r(K(\hat{c})) \setminus N_r(A)$. Notice that we have

$$\tilde{J}_q(g_1(\theta)) \geq \tilde{J}_q(g(\theta)) > \hat{c} - \varepsilon_2 \geq \hat{c} - \varepsilon_1, \tag{3.58}$$

hence $g_1(\theta) \in N_{r(\mu)}(A)$. Consider now the path $\gamma$ given first by moving $g_1(\theta)$ to $g_{3/2}(\theta)$ using $\psi(s, g_1(\theta))$ for $s \in [0,1]$ and then moving $g_{3/2}(\theta)$ to $g(\theta)$ using
\( \zeta_2(s, g_{3/2}(\theta)) \) for \( s \in [0, \sigma_2(g_{3/2}(\theta))] \). \( \gamma \) travels from \( N_{\frac{\mu}{64}}(A) \) to \( N_{\frac{\rho}{64}}(\mathcal{K}(\hat{c})) \setminus N_{\rho}(A) \).

Since \( \bar{r} < \frac{\mu}{64} \), we move from \( N_{\frac{\mu}{64}}(A) \) to \( N_{\frac{\rho}{64}}(\mathcal{K}(\hat{c})) \setminus N_{\rho}(A) \). Suppose that \( x \in N_{\frac{\mu}{64}}(\mathcal{K}(\hat{c})) \setminus N_{\rho}(A) \) and \( y \in N_{\frac{\rho}{64}}(A) \), then if \( u_x \in \mathcal{K}(\hat{c}) \setminus A \) and \( u_y \in A \) are such that \( \|u_x - x\| \) and \( \|u_y - y\| \) are both smaller than \( \frac{\mu}{64} \), we will have

\[
\mu \leq \|u_x - u_y\| \leq \|u_x - x\| + \|x - y\| + \|y - u_y\| \quad (3.59)
\]

Thus,

\[
\frac{31}{32} \mu \leq \|x - y\|. \quad (3.60)
\]

Since this is true for any \( x \in N_{\frac{\mu}{64}}(\mathcal{K}(\hat{c})) \setminus N_{\rho}(A) \) and any \( y \in N_{\frac{\rho}{64}}(A) \), we must have

\[
dist\left( N_{\frac{\mu}{64}}(\mathcal{K}(\hat{c})) \setminus N_{\rho}(A), N_{\frac{\rho}{64}}(A) \right) \geq \frac{31}{32} \mu. \quad (3.61)
\]

Thus, \( \gamma \) must travel at least \( \frac{31}{32} \mu \) through \( (\tilde{J}_{\tilde{q}})^{\tilde{r} + \frac{\varepsilon_1}{4}} \setminus N_{\frac{\mu}{64}}(\mathcal{K}(\hat{c})) \). Suppose now that \( [s_1, s_2] \subset [0, 1] \) and \( [t_1, t_2] \subset [0, \sigma_2(g_{3/2}(\theta))] \) are maximal intervals where \( \gamma \in (\tilde{J}_{\tilde{q}})^{\tilde{r} + \frac{\varepsilon_1}{4}} \setminus N_{\frac{\mu}{64}}(\mathcal{K}(\hat{c})) \) and for which \( \|\psi(s_1, g_1(\theta)) - \psi(s_2, g_1(\theta))\| \) and \( \|\zeta_2(t_1, g_{3/2}(\theta)) - \zeta_2(t_2, g_{3/2}(\theta))\| \) are largest. Notice that this implies that

\[
\|\psi(s_1, g_1(\theta)) - \psi(s_2, g_1(\theta))\| + \|\zeta_2(t_1, g_{3/2}(\theta)) - \zeta_2(t_2, g_{3/2}(\theta))\| \geq \frac{31}{32} \mu. \quad (3.62)
\]

Now, because of (3.54), we must have

\[
\tilde{J}_{\tilde{q}}(\psi(s_1, g_1(\theta))) - \tilde{J}_{\tilde{q}}(\psi(s_2, g_1(\theta))) = \int_{s_2}^{s_1} \frac{d}{ds} \tilde{J}_{\tilde{q}}(\psi(s, g_1(\theta))) ds \quad (3.63)
\]

Now, for \( s \in [s_1, s_2] \), we have \( \psi(s, g_1(\theta)) \in (\tilde{J}_{\tilde{q}})^{\tilde{r} + \frac{\varepsilon_1}{4}} \setminus N_{\frac{\mu}{64}}(\mathcal{K}(\hat{c})) \). Thus, in particular, we have \( \psi(s, g_1(\theta)) \in (\tilde{J}_{\tilde{q}})^{\tilde{r} + \frac{\varepsilon_1}{4}} \subset (\tilde{J}_{\tilde{q}})^{\tilde{r} + \frac{\varepsilon_1}{4}} \). Hence \( \hat{f}(\psi(s, g_1(\theta))) = 1 \) by
(3.52). In addition, since \( \rho_1 \leq \frac{\mu}{64} \) and \( \psi(s, g_1(\theta)) \not\in N_{\frac{\mu}{64}}(\mathcal{K}(\hat{c})) \), (3.51) implies that 
\[ \hat{\varphi}(\psi(s, g_1(\theta))) = 1. \] 
Thus, (3.63) implies that
\[ \tilde{J}_q(\psi(s_1, g_1(\theta))) = \frac{\varepsilon_1}{4} (s_2 - s_1). \] (3.64)

Now, since \( \psi(s, g_1(\theta)) \not\in N_{\frac{\mu}{64}}(\mathcal{K}(\hat{c})) \), Lemma 3.1.8 implies that 
\[ \|\tilde{J}'_q(\psi(s, g_1(\theta)))\| \geq \delta \left( \frac{\mu}{64} \right) \] 
for all \( s \in [s_1, s_2] \). Thus,
\[
\|\psi(s_1, g_1(\theta)) - \psi(s_2, g_1(\theta))\| 
\leq \int_{s_1}^{s_2} \frac{\varepsilon_1}{4\|J'_q(\psi(s, g_1(\theta)))\|} ds \] (3.65)
\[
\leq \frac{\varepsilon_1 (s_2 - s_1)}{4\delta \left( \frac{\mu}{64} \right)}. \]

Combining (3.64) and (3.65), we have
\[
\|\psi(s_1, g_1(\theta)) - \psi(s_2, g_1(\theta))\| 
\leq \frac{\tilde{J}_q(\psi(s_1, g_1(\theta))) - \tilde{J}_q(\psi(s_2, g_1(\theta)))}{\delta \left( \frac{\mu}{64} \right)}. \] (3.66)

Similarly, we will have
\[
\tilde{J}_q(\zeta_2(t_1, g_{3/2}(\theta))) - \tilde{J}_q(\zeta_2(t_2, g_{3/2}(\theta))) 
= \int_{t_1}^{t_2} \frac{d}{ds} \tilde{J}_q(\zeta_2(s, g_{3/2}(\theta))) ds \] (3.67)
\[
= \int_{t_1}^{t_2} \varepsilon \varphi_2(\zeta_2(s, g_{3/2}(\theta))) f_2(\zeta_2(s, g_{3/2}(\theta))) ds. \]

Because \( \zeta_2(s, g_{3/2}(\theta)) \in (\tilde{J}_q)^{\varepsilon_1 + \varepsilon_2} \setminus N_{\frac{\mu}{64}}(\mathcal{K}(\hat{c})) \) and \( \rho \leq \tilde{\rho} < \frac{\mu}{64} \), (3.21) and (3.23) imply
\[ \varphi_2(\zeta_2(s, g_{3/2}(\theta))) f_2(\zeta_2(s, g_{3/2}(\theta))) = 1 \]
for \( s \in [t_1, t_2] \), and so
\[
\tilde{J}_q(\zeta_2(t_1, g_{3/2}(\theta))) - \tilde{J}_q(\zeta_2(t_2, g_{3/2}(\theta))) = \varepsilon(t_2 - t_1). \] (3.68)
We also have
\[
\|\zeta_2(t_2, g_{3/2}(\theta)) - \zeta_2(t_1, g_{3/2}(\theta))\| \leq \int_{t_1}^{t_2} \frac{\bar{\varepsilon}}{\|J_q(\zeta_2(s, g_{3/2}(\theta)))\|_{E'}} \, ds \tag{3.69}
\]
\[
\leq \frac{\bar{\varepsilon}(t_2 - t_1)}{\delta(\frac{\mu}{64})}
\]
Combining (3.68) and (3.69), we will have
\[
\|\zeta_2(t_2, g_{3/2}(\theta)) - \zeta_2(t_1, g_{3/2}(\theta))\| \leq \tilde{J}_q(\zeta_2(t_1, g_{3/2}(\theta))) - \tilde{J}_q(\zeta_2(t_2, g_{3/2}(\theta))) \tag{3.70}
\]
Now, combining (3.62), (3.66) and (3.70), we must have
\[
\frac{31}{32} \mu \delta(\frac{\mu}{64}) \leq \tilde{J}_q(\zeta_2(t_1, g_{3/2}(\theta))) - \tilde{J}_q(\zeta_2(t_2, g_{3/2}(\theta)))
\]
\[
+ \tilde{J}_q(\psi(s_1, g_1(\theta))) - \tilde{J}_q(\psi(s_2, g_1(\theta)))
\]
\[
= \tilde{J}_q(\psi(s_1, g_1(\theta))) - \tilde{J}_q(\zeta_2(t_2, g_1(\theta)))
\]
\[
- \left( \tilde{J}_q(\psi(s_2, g_1(\theta))) - \tilde{J}_q(\zeta_2(t_1, g_1(\theta))) \right)
\]
\[
\leq \tilde{J}_q(g_1(\theta)) - \tilde{J}_q(g(\theta)) \leq \frac{\varepsilon_1}{4} + \varepsilon_2 \leq 2 \varepsilon_1. \tag{3.71}
\]
Now, by the definition of \(\varepsilon_1\) from (3.20) and recalling that \(\varepsilon_1\) is found by taking \(\bar{\varepsilon} = \tilde{\alpha}/2\) and \(r = (\mu/64)/2\), we must have
\[
\frac{31}{32} \mu \delta(\frac{\mu}{64}) \leq \varepsilon_1 \leq \frac{\mu}{128} \delta(\frac{\mu}{8^{2+64}}) \leq \frac{\mu \delta(\frac{\mu}{64})}{64}, \tag{3.72}
\]
which is impossible. Thus, we cannot have \(\gamma\) leaving \(N_{\mu/\varepsilon}(A)\), and so in fact we must have \(g(\theta) \in N_{\varepsilon}(A)\), and (2) holds.

Next, we describe a standard method of approximating elements of \(\hat{E}\) for which \(I\) is finite. Recall that if \(I(q) < \infty\), then \(q(-\infty) = \lim_{t \to -\infty} q(t)\) and \(q(\infty) = \lim_{t \to \infty} q(t)\) exist. For any such \(q\), we make the following definitions:
Definition 3.1.12. For any $\rho > 0$ and $q \in \hat{E}$ with $I(q) < \infty$, we define:

$$
\alpha(q, \rho) := \sup \{ t \in \mathbb{R} \mid q(s) \in B_{\rho}(q(-\infty)) \text{ for all } s < t \}
$$

$$
\omega(q, \rho) := \inf \{ t \in \mathbb{R} \mid q(s) \in B_{\rho}(q(\infty)) \text{ for all } s > t \}
$$

Notice that $\alpha(q, \rho) \to -\infty$ and $\omega(q, \rho) \to \infty$ as $\rho \to 0$, and $\alpha(q, \rho)$ is decreasing in $\rho$ and $\omega(q, \rho)$ is increasing in $\rho$. Moreover, for any $k \in \mathbb{Z}$, we have $\alpha(\tau_k q, \rho) = \alpha(q, \rho) + k$ and $\omega(\tau_k q, \rho) = \omega(q, \rho) + k$. Using $\alpha(q, \rho)$ and $\omega(q, \rho)$, we define an approximation of $q$.

Definition 3.1.13.

$$
q_{\rho}(t) := \begin{cases} 
q(-\infty) & \text{for } t \leq \alpha(q, \rho) - 1 \\
(\alpha(q, \rho) - t)q(-\infty) + (t - \alpha(q, \rho) + 1)q(\alpha(q, \rho)) & \text{for } \alpha(q, \rho) - 1 \leq t \leq \alpha(q, \rho) \\
q(t) & \text{for } t \in (\alpha(q, \rho), \omega(q, \rho)) \\
(t - \omega(q, \rho))q(\infty) + (\omega(q, \rho) + 1 - t)q(\omega(q, \rho)) & \text{for } \omega(q, \rho) \leq t \leq \omega(q, \rho) + 1 \\
q(\infty) & \text{for } t > \omega(q, \rho) + 1
\end{cases}
$$

Lemma 3.1.14. $\|q - q_{\rho}\|_{E} \to 0$ as $\rho \to 0$. 
Proof. We have, using Definition 3.1.12,
\[\|q - q_\rho\|^2 = \int_{-\infty}^{\alpha(q, \rho)-1} |\dot{q}(t)|^2 + |q(t) - q(-\infty)|^2 dt + \int_{\alpha(q, \rho)}^{\infty} |\dot{q}(t)|^2 + |q(t) - q(\infty)|^2 dt\]
\[+ \int_{\alpha(q, \rho)-1}^{\alpha(q, \rho)} |q(t) - q(-\infty) + (t - \alpha(q, \rho) + 1)(q(-\infty) - q(\alpha(q, \rho)))|^2 dt\]
\[+ \int_{\omega(q, \rho)}^{\alpha(q, \rho)-1} |\dot{q}(t) - (q(-\infty) - q(\alpha(q, \rho)))|^2 dt\]
\[+ \int_{\omega(q, \rho)}^{\omega(q, \rho)+1} |(t - \omega(q, \rho) - 1)(q(\omega(q, \rho)) - q(\infty))|^2 dt\]
\[+ \int_{\omega(q, \rho)}^{\omega(q, \rho)+1} |\dot{q}(t) + (q(\omega(q, \rho)) - q(\infty))|^2 dt.\]

But then we must have
\[\|q - q_\rho\|^2 \leq \int_{-\infty}^{\alpha(q, \rho)-1} |\dot{q}(t)|^2 + |q(t) - q(-\infty)|^2 dt + \int_{\omega(q, \rho)}^{\infty} |q(t) - q(\infty)|^2 dt + |\dot{q}(t)|^2 dt\]
\[+ 2 \int_{\alpha(q, \rho)-1}^{\alpha(q, \rho)} |q(t) - q(-\infty)|^2 + |\dot{q}(t)|^2 dt\]
\[+ 2 \int_{\omega(q, \rho)}^{\omega(q, \rho)-1} (1 + (t - \alpha(q, \rho) + 1)^2)|q(-\infty) - q(\alpha(q, \rho))|^2 dt\]
\[+ 2 \int_{\omega(q, \rho)}^{\omega(q, \rho)+1} |q(t) - q(\infty)|^2 + |\dot{q}(t)|^2 dt\]
\[+ 2 \int_{\omega(q, \rho)}^{\omega(q, \rho)+1} (1 + (t - \omega(q, \rho) - 1)^2)|q(\omega(q, \rho)) - q(\infty)|^2 dt\]
and so
\[\|q - q_\rho\|^2 \leq 2 \int_{-\infty}^{\alpha(q, \rho)} |q(t) - q(-\infty)|^2 + |\dot{q}(t)|^2 dt + 2 \int_{\omega(q, \rho)}^{\infty} |q(t) - q(\infty)|^2 + |\dot{q}(t)|^2 dt\]
\[+ 2 \int_{\alpha(q, \rho)-1}^{\alpha(q, \rho)} (1 + (t - \alpha(q, \rho) + 1)^2)|q(-\infty) - q(\alpha(q, \rho))|^2 dt\]
\[+ 2 \int_{\omega(q, \rho)}^{\omega(q, \rho)+1} (1 + (t - \omega(q, \rho) - 1)^2)|q(\omega(q, \rho)) - q(\infty)|^2 dt.\]

But by definition of \(\alpha(q, \rho)\) and \(\omega(q, \rho)\), we must have
\[|q(\alpha(q, \rho)) - q(-\infty)| \leq \rho \text{ and } |q(\omega(q, \rho)) - q(\infty)| \leq \rho, \quad (3.75)\]
and so
\[ \|q - q_\rho\|^2 \leq 2\|q - q(-\infty)\|^2_{W^{1,2}(-\infty, \alpha(q, \rho))} + 2\|q - q(\infty)\|^2_{W^{1,2}(\omega(q, \rho), \infty)} + 4\rho^2 + 4\rho^2. \]
(3.76)

Hence
\[ \|q - q_\rho\| \leq 2 \left( \|q - q(-\infty)\|_{W^{1,2}(-\infty, \alpha(q, \rho))} + \|q - q(\infty)\|_{W^{1,2}(\omega(q, \rho), \infty)} + 2\rho \right). \]
(3.77)

By Lemma 2.1.4 and the fact that \( \alpha(q, \rho) \to -\infty, \omega(q, \rho) \to \infty \) as \( \rho \to 0 \), (3.77) implies that \( \|q - q_\rho\| \to 0 \) as \( \rho \to 0 \).

Next, we state a corollary of Lemma 3.1.14 that will be useful to get uniform approximations.

**Corollary 3.1.15.** Suppose that \( a \leq \alpha(q, \rho) \) and \( \omega(q, \rho) \leq b \). Then, if we define
\[ q_{a,b}(t) := \begin{cases} 
q(-\infty) & \text{for } t \leq a - 1 \\
(a - t)q(-\infty) + (t - a + 1)q(a) & \text{for } a - 1 \leq t \leq a \\
q(t) & \text{for } a < t < b \\
(t - b)q(\infty) + (b + 1 - t)q(b) & \text{for } b \leq t \leq b + 1 \\
q(\infty) & \text{for } t > b + 1
\end{cases} \]
we will have
\[ \|q - q_{a,b}\| \leq 2 \left( \|q - q(-\infty)\|_{W^{1,2}(-\infty, \alpha(q, \rho))} + \|q - q(\infty)\|_{W^{1,2}(\omega(q, \rho), \infty)} + 2\rho \right). \]
(3.78)

**Proof.** If we replace \( \alpha(q, \rho) \) by \( a \) and \( \omega(q, \rho) \) by \( b \) in the proof of Lemma 3.1.14, we will get (3.78) by noting that since \( a \leq \alpha(q, \rho) \) and \( \omega(q, \rho) \leq b \), we will have
\[ \|q - q(-\infty)\|_{W^{1,2}(-\infty, a)} \leq \|q - q(-\infty)\|_{W^{1,2}(-\infty, \alpha(q, \rho))} \]
(3.79)
and
\[ \|q - q(\infty)\|_{W^{1,2}(\mathcal{D},\infty)} \leq \|q - q(\infty)\|_{W^{1,2}(\omega(q,\rho),\infty)}. \]  
(3.80)

Moreover, since \( a \leq \alpha(q, \rho) \) and \( \omega(q, \rho) \leq b \), the definition of \( \alpha(q, \rho) \) and \( \omega(q, \rho) \) implies that
\[ |q(a) - q(-\infty)| \leq \rho \] and \[ |q(b) - q(\infty)| \leq \rho. \]  
(3.81)

Combining these inequalities proves the corollary.

**Remark 3.1.16.** Notice that if \( \rho \) is sufficiently small that
\[ 2\left(\|q - q(-\infty)\|_{W^{1,2}(\infty,\alpha(q,\rho))} + \|q - q(\infty)\|_{W^{1,2}(\omega(q,\rho),\infty)} + 2\rho\right) < \frac{r}{128}, \]  
(3.82)

then Corollary 3.1.15 implies that for all \( a \leq \alpha(q, \rho) \) and all \( b \geq \omega(q, \rho) \), we will have
\[ \|q - q_{a,b}\| \leq \frac{r}{128}. \]  
(3.83)

Now, for every \( u \in A_1 \cup \{0\} \), (where \( A_1 \) is from Proposition 3.1.11) \((u + q_1)(-\infty) = 0\) and \((u + q_1)(\infty) = \xi\) and so there is a \( \rho_u(r) > 0 \) such that
\[ 2\left(\|u + q_1\|_{W^{1,2}(\infty,\alpha(u+q_1,\rho_u(r))}) + \|u + q_1 - \xi\|_{W^{1,2}(\omega(u+q_1,\rho_u(r)),\infty)} + 2\rho_u(r)\right) < \frac{r}{128}. \]  
(3.84)

Next, because \( A_1 \) is finite,
\[ \rho_{A_1}^* (r) := \inf \{\rho_u(r) \mid u \in A_1 \cup \{0\}\} > 0. \]  
(3.85)

Similarly, for every \( v \in A_2 \cup \{0\} \) (where \( A_2 \) is from Proposition 3.1.11), \((v + q_2)(-\infty) = \xi\) and \((v + q_2)(\infty) = 0\). Thus, there is a \( \rho_v(r) > 0 \) such that
\[ 2\left(\|v + q_2 - \xi\|_{W^{1,2}(\infty,\alpha(v+q_2,\rho_v(r))}) + \|v + q_2\|_{W^{1,2}(\omega(v+q_2,\rho_v(r)),\infty)} + 2\rho_v(r)\right) < \frac{r}{128}. \]  
(3.86)
Since $A_2$ is finite, we must have

$$\rho^*_{A_2}(r) := \inf \{ \rho_v(r) \mid v \in A_2 \cup \{0\} \} > 0. \quad (3.87)$$

Finally, we let

$$\rho^*(r) := \min \{ \rho^*_{A_1}(r), \rho^*_{A_2}(r) \}. \quad (3.88)$$

and then for any $\rho < \rho^*(r)$, we define

$$\alpha(A_1, \rho) := \inf \{ \alpha(u + q_1, \rho) \mid u \in A_1 \cup \{0\} \}$$

$$\omega(A_1, \rho) := \sup \{ \omega(u + q_1, \rho) \mid u \in A_1 \cup \{0\} \} \quad (3.89)$$

$$\alpha(A_2, \rho) := \inf \{ \alpha(v + q_2, \rho) \mid v \in A_2 \cup \{0\} \}$$

$$\omega(A_2, \rho) := \sup \{ \omega(v + q_2, \rho) \mid v \in A_2 \cup \{0\} \}.$$  

Notice that all of the numbers in (3.89) are finite, since $A_1, A_2$ are both finite. Moreover, notice that $\alpha(A_i, \rho) \searrow -\infty$ as $\rho \to 0$ for $i = 1, 2$ and $\omega(A_i, \rho) \nearrow \infty$ as $\rho \to 0$ for $i = 1, 2$.

**Lemma 3.1.17.** If $u \in A_1 \cup \{0\}, v \in A_2 \cup \{0\}$ and $\rho < \rho^*(r)$, then

$$\| (u + q_1) - (u + q_1)_{A_1, \rho} \| \leq \frac{r}{128} \text{ and }$$

$$\| (v + q_2) - (v + q_2)_{A_2, \rho} \| \leq \frac{r}{128}$$

where (using the notation of Corollary 3.1.15)

$$\rho_{A_1, \rho} := (u + q_1)_{A_1, \rho} \quad \text{and}$$

$$\rho_{A_2, \rho} := (v + q_2)_{A_2, \rho}.$$
Proof. If \( u \in A_1 \cup \{0\} \), then we must have

\[
\alpha(A_1, \rho) \leq \alpha(u + q_1, \rho) \leq \alpha(u + q_1, \rho^*_A(r)) \leq \alpha(u + q_1, \rho_u(r)) \text{ and }
\]

\[
\omega(u + q_1, \rho_u(r)) \leq \omega(u + q_1, \rho^*_A(r)) \leq \omega(u + q_1, \rho) \leq \omega(A_1, \rho).
\]

But then Corollary 3.1.15 implies that

\[
\|u + q_1 - (u + q_1)_{A_1, \rho}\| \leq 2\left(\|u + q_1\|_{W^{1,2}(-\infty, \alpha(u + q_1, \rho_u(r)))} + \|u + q_1 - \xi\|_{W^{1,2}(\omega(u + q_1, \rho_u(r)), \infty)} + 2\rho_u(r)\right) \leq \frac{r}{128}
\]

because of the choice of \( \rho_u(r) \). A similar argument proves the lemma for \( v \in A_2 \cup \{0\} \). \qed

For future reference, notice that

\[
(u + q_1)_{A_1, \rho}(t) := \begin{cases} 
0 & \text{for } t \leq \alpha(A_1, \rho) - 1 \\
(t - \alpha(A_1, \rho) + 1)(u + q_1)(\alpha(A_1, \rho)) & \text{for } \alpha(A_1, \rho) - 1 \leq t \leq \alpha(A_1, \rho) \\
(u + q_1)(t) & \text{for } \alpha(A_1, \rho) < t < \omega(A_1, \rho) \\
(t - \omega(A_1, \rho))\xi + (\omega(A_1, \rho) + 1 - t)(u + q_1)(\omega(A_1, \rho)) & \text{for } \omega(A_1, \rho) \leq t \leq \omega(A_1, \rho) + 1 \\
\xi & \text{for } t > \omega(A_1, \rho) + 1
\end{cases}
\]

(3.90)
and

\[
(v + q_2)_{A_2,\rho}(t) := \begin{cases} 
\xi \quad &\text{for } t \leq \alpha(A_2, \rho) - 1 \\
(\alpha(A_2, \rho) - t)\xi + (t - \alpha(A_2, \rho) + 1)(v + q_2)(\alpha(A_2, \rho)) \quad &\text{for } \alpha(A_2, \rho) - 1 \leq t \leq \alpha(A_2, \rho) \\
(v + q_2)(t) \quad &\text{for } \alpha(A_2, \rho) < t < \omega(A_2, \rho) \\
(\omega(A_2, \rho) + 1 - t)q(\omega(A_2, \rho)) \quad &\text{for } \omega(A_2, \rho) \leq t \leq \omega(A_2, \rho) + 1 \\
0 \quad &\text{for } t > \omega(A_2, \rho) + 1
\end{cases}
\]

(3.91)

Suppose now that \(u \in A_1, v \in A_2\). Then, by definition of \(\tilde{J}_{q_i} u + q_1 \) and \(v + q_2\) are solutions of (HS). For any \(\rho < \rho^*(r)\), there is a \(n(\rho, r) \in \mathbb{N}\) such that for all \(n \in \mathbb{N}\) with \(n \geq n(\rho, r)\), we have \(\omega(A_1, \rho) + 1 < \alpha(A_2, \rho) + n - 1\). Thus, if \(n > n(\rho, r)\), we define a “\(\rho\)-approximation” of the chain \((u + q_1, v + q_2)\) for any \(u \in A_1, v \in A_2\) as follows:

**Definition 3.1.18.**

\[
Q((u, v), \rho, k)(t) := \begin{cases} 
(u + q_1)_{A_1,\rho}(t) \quad &\text{for } t < \omega(A_1, \rho) + 1 \\
\xi \quad &\text{for } t \in [\omega(A_1, \rho) + 1, \alpha(A_2, \rho) + n - 1] \\
\tau_n(v + q_2)_{A_2,\rho}(t) \quad &\text{for } t > \alpha(A_2, \rho) + n - 1
\end{cases}
\]

The goal of these sections is to show that there are critical points of \(I\) that are close to the \(Q((u, v), \rho, k)\) defined in Definition 3.1.18. The following lemma will be important in getting bounds on \(\|I'(u)\|_{E'}\).

**Lemma 3.1.19.** For any finite subset \(U \subset E\), we have following dichotomy: either

(i) there is \(u \in \overline{N_r(U)}\) such that \(I'(u) = 0\) or
(ii) there is a $\delta(r, z) > 0$ such that $\|I'(u)\|_{E'} \geq \delta(r, z)$ for all $u \in \overline{N_r(U)}$.

Proof. Suppose that (i) and (ii) are both false. Since (ii) is false, there is a sequence $u_n \in \overline{N_r(U)}$ such that $\|I'(u_n)\|_{E'} \to 0$ as $n \to \infty$. By definition, this means there is a sequence $\{v_n\} \subset U$ such that $\|u_n - v_n\| \leq r$. Since $U$ is finite, passing to a subsequence, we will have $\|u_n - v\| \leq r$ for all $n$. Since $u_n$ is bounded, by passing to a subsequence, there is a $u$ such that $u_n$ converges to $u$ weakly. Since $\overline{B_r(v)}$ is convex and hence weakly closed, we know that $u \in \overline{B_r(v)}$. Next, we claim that $I'(u)\varphi = 0$ for all $\varphi \in C^\infty_c(\mathbb{R})$, which will imply that $u$ is a critical point for $I$, and thus contradict our assumption that (i) is false.

Suppose now that $\text{supp } \varphi \subset [a, c]$. Notice that since $u$ is the weak limit of the $u_n$, we know that $u_n \to u$ in $L^\infty(\mathbb{R})$ and $\dot{u}_n \to \dot{u}$ weakly in $L^2(\mathbb{R})$ as $n \to \infty$. Therefore, $u_n \to u$ in $L^\infty([a, c])$ as $n \to \infty$. Hence

$$\int_\mathbb{R} -\langle V_q(t, u_n(t)), \varphi(t) \rangle \, dt \to \int_\mathbb{R} -\langle V_q(t, u(t)), \varphi(t) \rangle \, dt \quad (3.92)$$

as $n \to \infty$. Because of the weak convergence, we also have

$$\int_\mathbb{R} \langle \dot{u}_n, \varphi \rangle \, dt \to \int_\mathbb{R} \langle \dot{u}, \varphi \rangle \, dt \quad (3.93)$$

as $n \to \infty$. Moreover, we have

$$I'(u_n)\varphi = \int_\mathbb{R} \langle \dot{u}_n, \varphi \rangle - \langle V_q(t, u_n), \varphi \rangle \, dt \to 0 \quad (3.94)$$

as $n \to \infty$. Combining (3.92) - (3.94), we must have

$$I'(u)\varphi = \int_\mathbb{R} \langle \dot{u}, \varphi \rangle - \langle V_q(t, u), \varphi \rangle \, dt = 0, \quad (3.95)$$

which shows our claim. \qed
Now, for any $\rho < \rho^*(r)$ and any $n > n(\rho, r)$, we define
\[
\mathcal{M}(A_1, A_2, \rho, n) := \{ z \in E \mid z = Q((u, v), \rho, n) \text{ for some } u \in A_1, v \in A_2 \}. \tag{3.96}
\]
Notice that for fixed $n, \rho$, $\mathcal{M}(A_1, A_2, \rho, n)$ is a finite set, since $A_1, A_2$ are finite. Then, we define
\[
\mathcal{N}(r, \rho) := \{ n \in \mathbb{N} \mid n > n(\rho, r) \text{ and } \overline{\mathcal{N}_r(\mathcal{M}(A_1, A_2, \rho, n))} \cap K(I) = \emptyset \}. \tag{3.97}
\]
Notice that by Lemma 3.1.19, there is a $\delta(r, \rho, n) > 0$ such that if $n \in \mathcal{N}(r, \rho)$, then
\[
\|I'(z)\| \geq \delta(r, \rho, n) \text{ for all } z \in \overline{\mathcal{N}_r(\mathcal{M}(A_1, A_2, \rho, n))} \tag{3.98}
\]
The following proposition shows that there is a bound away from zero on $\|I'(z)\|$ for $z$ in annular regions around $\mathcal{M}(A_1, A_2, \rho, n)$ for $n \in \mathcal{N}(2r, \rho)$ that depends only on $A_1, A_2$ and $r$. Recall that $\eta > 0$ is sufficiently small that $V(t, x)$ is strictly convex in $x$ for $x \in B_{\eta/2}(K(V))$, and there exist $0 < \beta_1 \leq \beta_2$ such that
\[
\beta_1|x - \xi|^2 \leq -V(t, x) \leq \beta_2|x - \xi|^2 \tag{3.99}
\]
for all $x \in B_{\eta/2}(K(V)), \xi \in K(V)$.

**Proposition 3.1.20.** If $r < \min\{\frac{\eta}{4}, \frac{\eta}{2C}\}$ (where $C$ is such that $\|u\|_{L^\infty} \leq C\|u\|$), then there is a $\delta(A_1, A_2, r) > 0$ such that $\|I'(z)\| \geq \delta(A_1, A_2, r)$ for all
\[
z \in \bigcup_{\rho \leq \rho^*(2r)} \left( \bigcup_{n \in \mathcal{N}(2r, \rho)} N_r(\mathcal{M}(A_1, A_2, \rho, n)) \setminus N_{r/8}(\mathcal{M}(A_1, A_2, \rho, n)) \right)
\]
Proof. If this is false, then there is a sequence $z_m$ with $\|I'(z_m)\| \to 0$ as $m \to \infty$, and
\[
z_m \in \bigcup_{\rho \leq \rho^*(2r)} \left( \bigcup_{n \in \mathcal{N}(2r, \rho)} N_r(\mathcal{M}(A_1, A_2, \rho, n)) \setminus N_{r/8}(\mathcal{M}(A_1, A_2, \rho, n)) \right) \tag{3.100}
\]
for all \( m \). Thus, there is a sequence \( \{ \rho_m \} \subset (0, \rho^*(2r)) \) such that

\[
z_m \in \bigcup_{n \in \mathbb{N}(2r, \rho_m)} N_r (\mathcal{M}(A_1, A_2, \rho_m, n)) \setminus N_{r/8} (\mathcal{M}(A_1, A_2, \rho_m, n)) . \tag{3.101}
\]

But then by definition, there is a sequence \( \{ n_m \} \subset \mathbb{N} \) with \( n_m \in \mathbb{N}(2r, \rho) \) such that

\[
z_m \in N_r (\mathcal{M}(A_1, A_2, \rho_m, n_m)) \setminus N_{r/8} (\mathcal{M}(A_1, A_2, \rho_m, n_m)) . \tag{3.102}
\]

This means that there must exist \( u_m \in A_1 \) and \( v_m \in A_2 \) such that

\[
\frac{r}{8} \leq \| z_m - Q((u_m, v_m), \rho_m, n_m) \| \leq r
\]

for all \( m \). Since \( A_1, A_2 \) are finite, passing to a subsequence, we may as well assume that \( u_m \equiv u \) and \( v_m \equiv v \) for all \( m \), \( u \in A_1, v \in A_2 \). For convenience, we take \( Q_m := Q((u, v), \rho_m, n_m) \). Thus,

\[
\frac{r}{8} \leq \| z_m - Q_m \| \leq r . \tag{3.103}
\]

The idea is to use our knowledge of the behavior of (PS) sequences to derive a contradiction to (3.103). First, we must show that \( I(z_m) \) is bounded. Suppose for the moment that \( I(Q_m) \) is bounded. For all large \( m \), we will have \( \| I'(z_m) \| \leq \frac{1}{2} \), and so (because \( I' \) is Lipschitz)

\[
\| I'(u) \| \leq \| I'(u) - I'(z_m) \| + \frac{1}{2} \\
\leq K \| u - z_m \| + \frac{1}{2} \tag{3.104}
\]

\[
\leq Kr + \frac{1}{2}
\]
for all \( u \in B_r(z_m) \). Thus,

\[
|I(z_m)| \leq |I(Q_m)| + \left| \int_0^1 \frac{d}{ds} I(sz_m + (1-s)Q_m) ds \right|
\]

\[
\leq |I(Q_m)| + \int_0^1 \|I'(sz_m + (1-s)Q_m)\| \|z_m - Q_m\| ds
\]

\[
|I(z_m)| \leq |I(Q_m)| + \left( Kr + \frac{1}{2} \right) r
\]

and so \( I(z_m) \) is bounded if \( I(Q_m) \) is bounded. To show that \( I(Q_m) \) is bounded, notice that by definition of \( Q_m \), we have

\[
I(Q_m) = I((u + q_1)_{A_1, \rho_m}) + I(\tau_{nm} (v + q_2)_{A_2, \rho_m})
\]

\[
= I((u + q_1)_{A_1, \rho_m}) + I((v + q_2)_{A_2, \rho_m})
\]

However, by (3.107), we know that

\[
\|(u + q_1)_{A_1, \rho_m} - (u + q_1)\| \leq \frac{2r}{128} = \frac{r}{64},
\]

since \( \rho_m < \rho^*(2r) \). Next, since \( I'(u + q_1) = 0 \), we have \( \|I'(q)\| \leq Kr \) for all \( q \) with \( \|q - (u + q_1)\| \leq r \). Therefore, we have

\[
|I((u + q_1)_{A_1, \rho_m})| \leq |I(u + q_1)|
\]

\[
+ \int_0^1 \|I'(s(u + q_1)_{A_1, \rho_m} + (1-s)(u + q_1))\|
\]

\[
\times \|(u + q_1)_{A_1, \rho_m} - (u + q_1)\| ds
\]

\[
\leq I(u + q_1) + Kr \frac{r}{64}.
\]

Hence, \( I((u + q_1)_{A_1, \rho_m}) \) is bounded independent of \( m \). Similarly, \( I((v + q_2)_{A_2, \rho_m}) \) is bounded.

Therefore, \( z_m \) is a (PS) sequence. We next show that there are exactly three maximal intervals with lengths unbounded in \( m \) and on which \( z_m(t) \in B_{\eta/2}(K(V)) \).
First, suppose there are only two such intervals: \((-\infty, \bar{t}_m)\) and \((\bar{t}_m, \infty)\). But then by Proposition 2.1.13, there is a sequence \(\{k_m\} \subset \mathbb{Z}\) and a \(u \in \mathcal{K}(I)\) such that (passing to a subsequence)

\[
\|z_m - \tau_{k_m}u\| \to 0
\]
as \(m \to \infty\). Since \(\tau_{k_m}u \in \mathcal{K}(I)\), (3.103) implies that for all large \(m\), we will have

\[
\|Q_m - \tau_{k_m}u\| \leq 2r. \tag{3.109}
\]

Since \(n_m \in \mathcal{N}(2r, \rho_m)\), by definition of \(\mathcal{N}(2r, \rho_m)\), we must have

\[
\mathcal{N}_{2r}(\mathcal{M}(A_1, A_2, \rho_m, n_m)) \cap \mathcal{K}(I) = \emptyset. \tag{3.110}
\]

which means that (3.109) is impossible. Thus, there are at least three intervals with lengths unbounded in \(m\) on which \(z_m(t) \in B_{\eta/2}(K(V))\). Before continuing, let us show that on a subsequence \(n_m \to \infty\) as \(m \to \infty\). We have two cases to consider:

(i) There is a subsequence with \(\rho_m \to 0\) as \(m \to \infty\), or

(ii) \(\rho_m \geq \hat{\rho} > 0\) for some \(\hat{\rho} > 0\) and all \(m\).

If (i) occurs, then \(\omega(A_1, \rho_m) \to \infty\) and \(\alpha(A_2, \rho_m) \to -\infty\) as \(m \to \infty\). Thus, in order for \(\omega(A_1, \rho_m) + 1 < \alpha(A_2, \rho_m) + n_m - 1\), we must have \(n_m \to \infty\). Next, suppose that (ii) occurs and \(n_m\) is bounded. Passing to a subsequence, we will have \(n_m = n\) for all large \(m\). Since \(\rho_m \geq \hat{\rho}\), we have

\[
\omega(A_1, \rho_m) \leq \omega(A_1, \hat{\rho}) \tag{3.111}
\]

and

\[
\alpha(A_2, \hat{\rho}) \leq \alpha(A_2, \rho_m)
\]
for all $m$. But then
\[ \alpha(A_2, \rho_m) + n - 1 - (\omega(A_1, \rho_m) + 1) \leq \alpha(A_2, \hat{\rho}) - \omega(A_1, \hat{\rho}) + n - 2 \tag{3.12} \]
for all $m$. Similarly, we will have
\[ \omega(A_1, \rho_m) - \alpha(A_2, \rho_m) \leq \omega(A_1, \hat{\rho}) - \alpha(A_1, \hat{\rho}) \tag{3.13} \]
\[ \omega(A_2, \rho_m) - \alpha(A_2, \rho_m) \leq \alpha(A_1, \hat{\rho}) - \alpha(A_2, \hat{\rho}) \]
for all $m$. But then
\[ \omega(A_2, \rho_m) + n + 1 - (\alpha(A_1, \rho_m) - 1) \text{ is bounded in } m. \tag{3.14} \]
Notice that if $t < \alpha(A_1, \rho_m) - 1$ or $t > \omega(A_2, \rho_m) + n + 1$, then $Q_m(t) = 0$. Therefore, for $t < \alpha(A_1, \rho_m) - 1$ or $t > \omega(A_2, \rho_m) + n + 1$, we will have
\[ |z_m(t)| = |z_m(t) - Q_m(t)| \leq \|z_m - Q_m\|_{L^\infty} \leq Cr < \frac{\eta}{2} \tag{3.15} \]
Therefore, the two maximal intervals on which $z_m(t) \in B_{\eta/2}(K(V))$ with infinite length, i.e. $(-\infty, t_m)$ and $\bar{t}_m, \infty)$, must contain $(-\infty, \alpha(A_1, \rho_m) - 1)$ and $(\omega(A_2, \rho_m) + n + 1, \infty)$ respectively. Since there has to be another maximal interval on which $z_m(t) \in B_{\eta/2}(K(V))$ and whose length is unbounded in $m$, it must be a subset of $(\alpha(A_1, \rho_m) - 1, \omega(A_2, \rho_m) + n + 1)$, and by (3.14), the length of this interval is bounded in $m$. Thus, if case (ii) occurs, we must have $n_m \to \infty$ as $m \to \infty$.

So far, we know that there are at least three maximal intervals on which $z_m(t) \in B_{\eta/2}(K(V))$ whose lengths are unbounded in $m$. Next, notice that if
\[ t \in (-\infty, \alpha(u + q_1, \eta/2)) \cup (\omega(u + q_1, \eta/2), \alpha(v + q_2, \eta/2) + n_m) \]
\[ \cup (\omega(v + q_2, \eta/2) + n_m, \infty) \tag{3.16} \]
then \( Q_m(t) \in \overline{B_{\eta/2}(K(V))} \). To see this, notice that if \( t < \alpha(u + q_1, \eta/2) \), then by definition of \( \alpha(u + q_1, \eta/2) \), we have \( (u + q_1)(t) \in \overline{B_{\eta/2}(0)} \). But \( \alpha(u + q_1, \eta/2) < \omega(u + q_1, \eta/2) \leq \omega(A_1, \rho_m) \), and so \( Q_m(t) \) is either equal to \( (u + q_1)(t) \), 0, or a convex combination of these. Thus,

\[
Q_m(t) \in \overline{B_{\eta/2}(0)} \text{ for } t < \alpha(u + q_1, \eta/2). \tag{3.117}
\]

Similarly, we will have

\[
Q_m(t) \in \overline{B_{\eta/2}(0)} \text{ for } t > \omega(v + q_2, \eta/2) + n_m. \tag{3.118}
\]

Finally, suppose \( t \in (\omega(u + q_1, \eta/2), \alpha(v + q_2, \eta/2) + n_m) \). Notice that since \( \rho_m < \rho^*(2r) \leq \eta/2 \), we have \( \omega(u + q_1, \eta/2) \leq \omega(u + q_1, \rho_m) \leq \omega(A_1, \rho_m) \). Similarly, \( \alpha(A_2, \rho_m) \leq \alpha(v + q_2, \rho_m) \leq \alpha(v + q_2, \eta/2) \). We have a number of possibilities:

(i) \( Q_m(t) = (u + q_1)(t) \) if \( \omega(u + q_1, \eta/2) < t < \omega(A_1, \rho_m) \)

(ii) \( Q_m(t) \) is a convex combination of \( (u + q_1)(\omega(A_1, \rho_m)) \) and \( \xi \) if

\[
t \in [\omega(A_1, \rho_m), \omega(A_1, \rho_m) + 1]
\]

(iii) \( Q_m(t) = \xi \) for \( t \in (\omega(A_1, \rho_m) + 1, \alpha(A_2, \rho_m) + n_m - 1) \)

(iv) \( Q_m(t) \) is a convex combination of \( \tau_{nm}(v + q_2)(\alpha(A_2, \rho_m) + n_m) \) and \( \xi \) if \( t \in [\alpha(A_2, \rho_m) + n_m - 1, \alpha(A_2, \rho_m) + n_m] \)

(v) \( Q_m(t) = \tau_{nm}(v + q_2)(t) \) if \( \alpha(A_2, \rho_m) + n_m < t < \alpha(v + q_2, \eta/2) + n_m \)

In case (i), \( Q_m(t) = (u + q_1)(t) \in \overline{B_{\eta/2}(\xi)} \), since \( t > \omega(u + q_1, \eta/2) \). Similarly, in case (v), \( Q_m(t) \in \overline{B_{\eta/2}(\xi)} \). In case (ii), since \( \omega(u + q_1, \rho_m) < \omega(A_1, \rho_m) \) and \( \rho_m < \eta/2 \),
we will have $(u + q_1)(\omega(A_1, \rho_m)) \in \overline{B_{\rho_m}(\xi)} \subset B_{\eta/2}(\xi)$. A similar argument holds for case (iv). In any case, we must then have

$$Q_m(t) \in \overline{B_{\eta/2}(\xi)} \text{ for } t \in (\omega(u + q_1, \eta/2), \alpha(v + q_2, \eta/2) + n_m).$$

(3.119)

Since $n_m \to \infty$, the length of this last interval is unbounded in $m$. If there are four or more maximal intervals on which $z_m(t) \in B_{\eta/2}(K(V))$ with lengths unbounded in $m$, then there are at least four maximal intervals on which $z_m(t) \in B_{\eta}(K(V))$.

Since

$$\|z_m - Q_m\|_{L^\infty} \leq C\|z_m - Q_m\| \leq Cr < \frac{\eta}{2},$$

(3.120)

by the choice of $r$, there must exist maximal intervals on which $z_m(t) \in B_{\eta}(K(V))$ containing $(-\infty, \alpha(u + q_1, \eta/2)), (\omega(u + q_1, \eta/2), \alpha(v + q_2, \eta/2) + n_m)$ and $(\omega(v + q_2, \eta/2) + n_m, \infty)$. If there were a fourth such interval with length unbounded in $m$, then it must be contained in either $[\alpha(u + q_1, \eta/2), \omega(u + q_1, \eta/2)]$ or $[\alpha(v + q_2, \eta/2) + n_m, \omega(v + q_2, \eta/2) + n_m]$. But these intervals are bounded in length. Therefore, there can be at most three maximal intervals on which $z_m(t) \in B_{\eta}(K(V))$ whose lengths are unbounded in $m$, and so there are exactly three maximal intervals on which $z_m(t) \in B_{\eta/2}(K(V))$ whose lengths are unbounded in $m$. For convenience, we label them

$$(-\infty, \bar{t}_{m_1}), (\bar{t}_{m_2}, \bar{t}_{m_2}), \text{ and } (\bar{t}_{m_3}, \infty).$$

(3.121)

Observe that for $t$ in the intervals with infinite length, we must have $z_m(t) \in B_{\eta/2}(0)$, while for $t \in (\bar{t}_{m_2}, \bar{t}_{m_2})$, we will have $z_m(t) \in B_{\eta/2}(\xi)$.

Next, notice that if $t \in (\omega(A_1, \rho_m) + 1, \alpha(A_2, \rho_m) + n_m - 1)$, then $Q_m(t) = \xi$. 


Since \( \| z_m - Q_m \|_{L^\infty} < \eta/2 \), we must have
\[
(\omega(A_1, \rho_m) + 1, \alpha(A_2, \rho_m) + n_m - 1) \subset (t_{m_2}^*, \bar{t}_{m_2}^*).
\]
We now claim that if
\[
t^*_m := \frac{\omega(A_1, \rho_m) + 1 + \alpha(A_2, \rho_m) + n_m - 1}{2},
\]
then
\[
t^*_m - t_{m_2}^* \to \infty \quad \text{and} \quad \bar{t}_{m_2}^* - t^*_m \to \infty
\]
as \( m \to \infty \). Suppose that (3.124) is false. Then, passing to a subsequence, we will have for either \( t^*_m - t_{m_2}^* \) or \( \bar{t}_{m_2}^* - t^*_m \) bounded. Let us consider the case where \( t^*_m - t_{m_2}^* \) is bounded. Thus, there is a \( C > 0 \) such that
\[
C > t^*_m - t_{m_2}^* = \frac{1}{2}(\omega(A_1, \rho_m) + 1 - t_{m_2}^*) + \frac{1}{2}(\alpha(A_2, \rho_m) + n_m - 1 - t_{m_2}^*) > 0.
\]
Since \( t_{m_2}^* < \omega(A_1, \rho_m) + 1 < \alpha(A_2, \rho_m) + n_m - 1 \), (3.125) implies that
\[
\omega(A_1, \rho_m) + 1 - t_{m_2}^* \quad \text{and} \quad \alpha(A_2, \rho_m) + n_m - 1 - t_{m_2}^*
\]
are both bounded in \( m \). Notice that if \( \rho_m \geq \hat{\rho} > 0 \) for all \( m \), then we must have
\[
\omega(A_1, \rho_m) \leq \omega(A_1, \hat{\rho}) \quad \text{and} \quad \alpha(A_2, \hat{\rho}) \leq \alpha(A_2, \rho_m).
\]
Hence
\[
(\alpha(A_2, \hat{\rho}) + n_m - 1) - (\omega(A_1, \hat{\rho}) + 1)
\]
\[
\leq (\alpha(A_2, \rho_m) + n_m - 1) - (\omega(A_1, \rho_m) + 1) \to \infty
\]
as $m \to \infty$, since $n_m \to \infty$. However,

\[
(\alpha(A_2, \rho_m) + n_m - 1) - (\omega(A_1, \rho_m) + 1) = (\alpha(A_2, \rho_m) + n_m - 1 - t_{m_2}) - (\omega(A_1, \rho_m) + 1 - t_{m_2}) \tag{3.129}
\]

and both of the terms on the right in (3.129) are bounded, contradicting (3.128). Thus, passing to a subsequence, we may assume that $\rho_m \to 0$ as $m \to \infty$. But then

\[
(\omega(A_1, \rho_m) + 1) - \omega(u + q_1, \eta/2) \to \infty \tag{3.130}
\]

as $m \to \infty$. Now

\[
t_{m_2} - (\omega(u + q_1, \eta/2)) = (t_{m_2} - \omega(A_1, \rho_m)) + (\omega(A_1, \rho_m) - \omega(u + q_1, \eta/2)). \tag{3.131}
\]

Since the second term on the right tends to $\infty$ as $m \to \infty$ and we are assuming that the first term on the right is bounded, (3.131) implies that

\[
t_{m_2} - \omega(u + q_1, \eta/2) \to \infty \tag{3.132}
\]

as $m \to \infty$. Thus, the length of the interval $(\omega(u + q_1, \eta/2), t_{m_2})$ is unbounded in $m$. Since $\omega(u + q_1, \eta/2) \leq \omega(A_1, \rho_m)$, if $t \in (\omega(u + q_1, \eta/2), t_{m_2})$, then $Q_m(t) = (u + q_1)(t) \in \overline{B_{\eta/2}(\xi)}$. Thus, (3.120) implies that

\[
\text{if } t \in (\omega(u + q_1, \eta/2), t_{m_2}), \text{ then } z_m(t) \in B_\eta(\xi). \tag{3.133}
\]
Notice that

\[
(\omega(u + q_1, \eta/2), t_m) = \left\{ t \in (\omega(u + q_1, \eta/2), t_m) \mid z_m(t) \in B_\eta(\xi) \setminus B_{\eta/2}(\xi) \right\}
\]

\[
\bigcup \left\{ t \in (\omega(u + q_1, \eta/2), t_m) \mid z_m(t) \in B_{\eta/2}(\xi) \right\}
\]

\[
= A_m \sqcup B_m
\]

where \( \sqcup \) denotes a disjoint union. Now, we know that

\[
I(z_m) \geq \int_{A_m} \frac{1}{2} \dot{z}_m^2 - V(t, z_m) dt \geq \left( \inf_{t \in [0,1], x \not\in B_{\eta/2}(K(V))} -V(t, x) \right) |A_m|, \quad (3.135)
\]

so \( |A_m| \) is bounded. Therefore, we must have \( |B_m| \to \infty \) as \( m \to \infty \). Notice that

\( B_m \) is an open subset of \( \mathbb{R} \), hence \( B_m \) is the disjoint union of open intervals. We divide these intervals into two groups: those for which \( z_m(t) \) avoids \( B_{\eta/4}(\xi) \) and those for which \( z_m(t) \) intersects \( B_{\eta/4}(\xi) \). More precisely, we have

\[
B_m = \left( \bigcup_{i=1}^{j_E(m)} (a_i^m, b_i^m) \right) \sqcup \left( \bigcup_{i=1}^{j_H(m)} (c_i^m, d_i^m) \right)
\]

where the \((a_i^m, b_i^m)\) are maximal intervals for which \( z_m(t) \not\in B_{\eta/4}(\xi) \) for all \( t \in \sqcup_{i=1}^{j_E(m)} (a_i^m, b_i^m) \) and there is a \( t_i^m \in (c_i^m, d_i^m) \) such that \( z_m(t_i^m) \in B_{\eta/4}(\xi) \). We have

\[
I(z_m) \geq \sum_{i=1}^{j_E(m)} \int_{a_i^m}^{b_i^m} \frac{1}{2} |\dot{z}_m|^2 - V(t, z_m) dt
\]

\[
\geq \sum_{i=1}^{j_E(m)} \int_{a_i^m}^{b_i^m} -V(t, z_m) dt \quad (3.137)
\]

\[
\geq \left( \inf_{t \in [0,1], x \not\in B_{\eta/4}(K(V))} -V(t, x) \right) \sum_{i=1}^{j_E(m)} (b_i^m - a_i^m).
\]

Thus, \( \sum_{i=1}^{j_E(m)} (b_i^m - a_i^m) \) is bounded as \( m \to \infty \). Therefore, in order for \( |B_m| \to \infty \)
as \( m \to \infty \), we must have
\[
\sum_{i=1}^{j_H(m)} (d_i^m - c_i^m) \to \infty
\] (3.138)
as \( m \to \infty \). Next, notice that \((\omega(u + q_1, \eta/2), t_{m_2})\) is disjoint from \((t_{m_2}, \bar{t}_{m_2})\), and so there is a \( \tilde{C} \) such that \( d_i^m - c_i^m \leq \tilde{C} \) for all \( i, m \). (This follows from the fact that there are exactly three maximal intervals on which \( z_m(t) \in B_{\eta/2}(K(V)) \) with lengths unbounded in \( m \).) Thus, we must have
\[
\sum_{i=1}^{j_H(m)} (d_i^m - c_i^m) \leq \tilde{C} j_H(m),
\] (3.139)
and so \( j_H(m) \to \infty \) as \( m \to \infty \). But by Lemma 2.1.10, we have
\[
\int_{c_i^m}^{d_i^m} \frac{1}{2} |\dot{z}_m|^2 - V(t, z_m) \geq \frac{\eta}{4} \sqrt{2\alpha \left( \frac{\eta}{4} \right)},
\] (3.140)
since on \((c_i^m, d_i^m), z_m\) travels from \( \partial B_{\eta/2}(\xi) \) at the endpoints to \( B_{\eta/4}(\xi) \) at some point \( t^m_i \in (c_i^m, d_i^m) \). But we must have
\[
I(z_m) \geq \sum_{i=1}^{j_H(m)} \int_{c_i^m}^{d_i^m} \frac{1}{2} |\dot{z}_m|^2 - V(t, z_m)dt \geq \left( \frac{\eta}{4} \sqrt{2\alpha \left( \frac{\eta}{4} \right)} \right) j_H(m).
\] (3.141)
But this is impossible, since \( j_H(m) \to \infty \) and \( I(z_m) \) is bounded. Therefore, we cannot have \( t^*_m - t_{m_2} \) bounded. A similar argument shows that \( \bar{t}_{m_2} - t^*_m \) is not bounded. Thus (3.124) is proved.

Therefore, as in Proposition 2.1.20, if we take
\[
z^1_m(t) := \begin{cases} 
z_m(t) & \text{for } t < t^*_m \\
(t^*_m + 1) - t)z_m(t^*_m) + (t - t^*_m)\xi & \text{for } t^*_m \leq t \leq t^*_m + 1 \\
\xi & \text{for } t > t^*_m + 1
\end{cases}
\] (3.142)
and

\[
Z_m^2(t) := \begin{cases} 
\xi & \text{for } t < t_m^* - 1 \\
(t - (t_m^* - 1))Z_m(t_m^*) + (t_m^* - t)\xi & \text{for } t_m^* - 1 \leq t \leq t_m^* \\
Z_m(t) & \text{for } t > t_m^*
\end{cases}
\]  

(3.143)

then each \(Z_m^i\) is a (PS) sequence for \(i = 1, 2\) with exactly two maximal intervals on which \(Z_m^i(t) \in B_{\eta/2}(K(V))\) with lengths unbounded in \(m\). Thus, by Proposition 2.1.13, there exist sequence \(\{k_m^i\} \subset \mathbb{Z}\) and \(u_i \in K(I)\) such that

\[
\|Z_m^i - \tau_{k_m^i} u_i\| \to 0
\]

(3.144)
as \(m \to \infty\). We now show that \(\tau_{k_m^1} u_1 \equiv (u + q_1)\) and \(\tau_{k_m^2} u_2 = \tau_{n_m} (v + q_2)\). To do this, we show that

\[
\|\tau_{k_m^1} u_1 - (u + q_1)\| < \mu \text{ and } \|\tau_{k_m^2} u_2 - \tau_{n_m} (v + q_2)\| < \mu.
\]

(3.145)

Assuming (3.145), we will have for example that

\[
\|(\tau_{k_m^1} u_1 - q_1) - u\| < \mu
\]

(3.146)

But \(\tilde{J}_{q_1}(\tau_{k_m^1} u_1 - q_1) = I'(\tau_{k_m^1} u_1) = 0\), and thus by Corollary 3.1.6 and (3.146), we must have

\[
\tau_{k_m^1} u_1 - q_1 = u.
\]

(3.147)
as elements of \(E\), which implies our claim.

Since \(\rho_m \leq \rho^*(2r)\), we will have

\[
\|\tau_{k_m^1} u_1 - (u + q_1)\| \leq \|\tau_{k_m^1} u_1 - Z_m^1\| + \|Z_m^1 - (u + q_1)_{A_1, \rho_m}\|
\]

\[
+ \|(u + q_1)_{A_1, \rho_m} - (u + q_1)\|
\]

\[
\leq \varepsilon_m^1 + \|Z_m^1 - (u + q_1)_{A_1, \rho_m}\| + \frac{2r}{128}
\]

(3.148)
where \( \varepsilon_m^1 \rightarrow 0 \) as \( m \rightarrow \infty \). Next, notice that since \( t_m^* > \omega(A_1, \rho_m) \), if \( t > t_m^* \), we know that \((u + q_1)_{A_1, \rho_m}(t) = \xi \). Thus,

\[
\|z_m^1 - (u + q_1)_{A_1, \rho_m}\|^2 = \|z_m^1 - \xi\|^2_{W^{1,2}(t_m^*, t_m^* + 1)} + \|z_m - (u + q_1)_{A_1, \rho_m}\|^2_{W^{1,2}(-\infty, t_m^*)}.
\]

(3.149)

But, for \( t \in (-\infty, t_m^*) \), we claim that \((u + q_1)_{A_1, \rho_m}(t) = Q_m(t) \). This is clear for \( t \in (-\infty, \omega(A_1, \rho_m)+1) \). If \( t \in [\omega(A_1, \rho_m)+1, t_m^*) \), then we have \((u + q_1)_{A_1, \rho_m}(t) = \xi \).

Since \( t_m^* < \alpha(A_2, \rho_m) + n_m - 1 \), we will have \( Q_m(t) = \xi \) for \( t \in [\omega(A_1, \rho_m), t_m^*) \).

Thus,

\[
\|z_m^1 - (u + q_1)_{A_1, \rho_m}\|^2 \leq \|z_m^1 - \xi\|^2_{W^{1,2}(t_m^*, t_m^* + 1)} + \|z_m + Q_m\|^2
\leq \|z_m^1 - \xi\|^2_{W^{1,2}(t_m^*, t_m^* + 1)} + r^2.
\]

(3.150)

Now, for \( t \in (t_m^*, t_m^* + 1) \), we have

\[
z_m^1(t) - \xi = (t_m^* + 1 - t)(z_m(t_m^*) - \xi)
\]

(3.151)

and so

\[
\|z_m^1 - \xi\|^2_{W^{1,2}(t_m^*, t_m^* + 1)} = \int_{t_m^*}^{t_m^* + 1} \left((t_m^* + 1 - t)^2 + 1\right)|z_m(t_m^*) - \xi|^2\,dt
\leq 2|z_m(t_m^*) - \xi| \rightarrow 0
\]

(3.152)

as \( m \rightarrow \infty \) by Lemma 2.1.19 and (3.124). Thus, for all sufficiently large \( m \), (3.150) and (3.152) imply that

\[
\|z_m^1 - (u + q_1)_{A_1, \rho_m}\|^2 \leq 4r^2.
\]

(3.153)

Combining (3.148) and (3.153), we will have

\[
\|\tau_{k_n} u_1 - (u + q_1)\| \leq \varepsilon_m^1 + 2r + \frac{r}{64} < 2\frac{\mu}{64} + 2\frac{\mu}{4} < \mu
\]

(3.154)
and so $\tau_{k_1} u_1 = u + q_1$. A similar argument shows that $\tau_{k_2} u_2 = \tau_n(v + q_2)$. Now, let

$$q_m(t) := \begin{cases} (u + q_1)_{A_1, \rho_m}(t) & \text{for } t < t^* \\ \tau_n(v + q_2)_{A_2, \rho_m}(t) & \text{for } t > t^* \end{cases}$$

(3.155)

Then, we claim that $q_m = Q_m$. This is clear if $t < \omega(A_1, \rho_m) + 1$ or $t > \alpha(A_2, \rho_m) + n_m - 1$. Suppose then that $t \in [\omega(A_1, \rho_m) + 1, \alpha(A_2, \rho_m) + n_m - 1]$. Then, $Q_m(t) = \xi$, and $(u + q_1)_{A_1, \rho_m}(t) = \xi = \tau_n(v + q_2)_{A_2, \rho_m}(t)$. Thus, $q_m(t) = \xi$ in this interval, and so $q_m = Q_m$. But then we have

$$\left(\frac{r}{8}\right)^2 \leq \|z_m - Q_m\|^2 = \|z_m - q_m\|^2_{W^{1,2}(-\infty, t^*_m)} + \|z_m - q_m\|^2_{W^{1,2}(t^*_m, \infty)}$$

$$\leq (\|z_m - (u + q_1)\|_{W^{1,2}(-\infty, t^*_m)} + \|(u + q_1) - (u + q_1)_{A_1, \rho_m}\|_{W^{1,2}(-\infty, t^*_m)})^2$$

$$+ (\|z_m - \tau_n(v + q_2)\|_{W^{1,2}(t^*_m, \infty)} + \|\tau_n(v + q_2) - \tau_n(v + q_2)_{A_2, \rho_m}\|_{W^{1,2}(t^*_m, \infty)})^2$$

(3.156)

$$\leq (\|z_1^1 - \tau_{k_1} u_1\| + \frac{r}{64})^2 + \left(\|z_2^1 - \tau_{k_2} u_2\| + \frac{r}{64}\right)^2$$

since $\tau_{k_1} u_1 = (u + q_1)$, $\tau_{k_2} u_2 = \tau_n(v + q_2)$, $\rho_m \leq \rho^*(2r)$, $z^1_m(t) = z_m(t)$ for $t < t^*_m$ and $z^2_m(t) = z_m(t)$ for $t > t^*_m$. Thus, we must have for all suitably large $m$

$$\frac{r}{8} \leq \varepsilon^1_m + \varepsilon^2_m + \frac{\mu}{32} \leq \frac{r}{16}$$

(3.157)

which is impossible. Thus, we cannot have $I'(z_m) \to 0$ and satisfy (3.103), and the Proposition is proved. 

We would like to find critical points $u$ of $I$ that are close to chains $(q_1 + u_1, q_2 + u_2)$ of heteroclinic solutions with $u_i \in A_i$ for $i = 1, 2$, in the sense that $u$ is near $q_1 + u_1$ for a semi-infinite interval, stays near $\xi$ for some amount of time, and then is near
\( \tau_k(q_2 + u_2) \) on a semi-infinite interval, where the solutions travels back to 0. To do this, we will define a minimax level, which will approximate the critical value we expect for a solution of (HS) that shadows \((q_1 + u_1, q_2 + u_2)\). With this in mind, we need to define some approximations. Before we do that, we show that \( \hat{c}_i \) is independent of the end points of paths in \( \Gamma_i \).

**Lemma 3.1.21.** For \( d > 0 \), let

\[
\hat{c}_i(d) := \inf_{h \in \Gamma_{i,d}} \max_{s \in [0,1]} \tilde{J}_{q_i}(h(s))
\]

where

\[
\Gamma_{i,d} := \{ h \in C([0,1], E) \mid h(0) \in B_d(0), h(1) \in B_d(\tau q_i - q_i) \}
\]

Then there is a \( d_0 > 0 \) such that for all \( d \leq d_0 \), we have \( \hat{c}_i(d) = \hat{c}_i \).

**Proof.** Notice that we have \( \hat{c}_{i,d} \leq \hat{c}_i \). We may assume that \( d \) is sufficiently small that \( B_d(0) \cap B_d(\tau_1 q_i - q_i) = \emptyset \). Since \( \hat{c}_i > 0 \) and \( \tilde{J}_{q_i}(0) = \tilde{J}_{q_i}(\tau q_i - q_i) = 0 \), there is a \( d_0 \) such that if \( u \in B_{d_0}(0) \cup B_{d_0}(\tau q_i - q_i) \), then \( \tilde{J}_{q_i}(u) < \hat{c}_i/2 \). Suppose now that \( d < d_0 \). Then, for any \( h \in \Gamma_{i,d} \), we may concatenate straight line segments connecting \( h(0) \) to 0 and \( h(1) \) to \( \tau q_i - q_i \) to get an element \( \tilde{h} \in \Gamma_i \). Because \( d < d_0 \), \( \tilde{J}_{q_i}(\tilde{h}(s)) < \hat{c}_i \) for all \( s \) with \( \tilde{h}(s) \in B_d(0) \cup B_d(\tau q_i - q_i) \). Thus, the maximum of \( \tilde{J}_{q_i}(\tilde{h}(s)) \) occurs when \( \tilde{h}(s) \in h([0,1]) \). Hence

\[
\hat{c}_i \leq \max_{s \in [0,1]} \tilde{J}_{q_i}(\tilde{h}(s)) = \max_{s \in [0,1]} \tilde{J}_{q_i}(h(s)).
\]

Since this is true for every \( h \in \Gamma_{i,d} \), we must have

\[
\hat{c}_i \leq \inf_{h \in \Gamma_{i,d}} \max_{s \in [0,1]} \tilde{J}_{q_i}(h(s)) = \hat{c}_{i,d},
\]

which finishes the proof. \( \square \)
Now, we introduce a set that we will use to define a minimax value, which will be instrumental in our gluing procedure.

**Definition 3.1.22.**

$$\Gamma(d) := \{ g \in C([0, 1]^2, E) \mid g(\theta_1, \theta_2) \text{ satisfies (i) and (ii)} \}$$

where

(i) There exist real numbers $$a_1 < a_2$$ (which depend on $$g$$) such that

$$\text{for } t \in (a_1, a_2), g(\theta)(t) = \xi \text{ for all } \theta \in [0, 1]^2.$$ 

Next, let

$$g_1(\theta_1, \theta_2)(t) := \begin{cases} 
g(\theta_1, \theta_2)(t) & \text{for } t < a_1 \\
\xi & \text{for } t > a_1 \end{cases}$$  \hspace{1cm} (3.158)

and

$$g_2(\theta_1, \theta_2)(t) := \begin{cases} 
\xi & \text{for } t < a_2 \\
g(\theta_1, \theta_2)(t) & \text{for } t > a_2 \end{cases}$$  \hspace{1cm} (3.159)

(ii) There is a $$k \in \mathbb{Z}$$ such that

$$\|g_1(0, \theta_2) - q_1\|_E < d \text{ for all } \theta_2 \in [0, 1].$$

$$\|g_1(1, \theta_2) - \tau_1 q_1\|_E < d \text{ for all } \theta_1 \in [0, 1]$$

$$\|g_2(\theta_1, 0) - \tau_k q_2\|_E < d \text{ for all } \theta_2 \in [0, 1]$$

$$\|g_2(\theta_1, 1) - \tau_k \tau_1 q_2\|_E < d \text{ for all } \theta_2 \in [0, 1]$$

Next, let

$$c(d) := \inf_{h \in \Gamma(d)} \max_{\theta \in [0, 1]^2} I(h(\theta))$$
Next, we collect some important facts about elements of \( \Gamma(d) \). Notice that if \( g \in \Gamma(d) \), then

\[
I(g(\theta)) = \int_{-\infty}^{a_1} \left( \frac{1}{2} |\dot{g}(\theta)(t)|^2 - V(t, g(\theta)(t)) \right) dt + \int_{a_2}^{\infty} \left( \frac{1}{2} |\dot{g}(\theta)(t)|^2 - V(t, g(\theta)(t)) \right) dt
\]

\[
= \int_{-\infty}^{a_1} \left( \frac{1}{2} |\dot{g}_1(\theta)(t)|^2 - V(t, g_1(\theta)(t)) \right) dt + \int_{a_2}^{\infty} \left( \frac{1}{2} |\dot{g}_2(\theta)(t)|^2 - V(t, g_2(\theta)(t)) \right) dt
\]

\[
= I(g_1(\theta)) + I(g_2(\theta)).
\]

With these definitions, (3.160) becomes

\[
I(g(\theta_1, \theta_2)) = I(g_1(\theta)) + I(g_2(\theta))
\]

\[
= I(g_1(\theta)) + I(\tau_{-k}g_2(\theta))
\]

\[
= I((g_1(\theta) - q_1) + q_1) + I((\tau_{-k}g_2(\theta) - q_2) + q_2)
\]

\[
= \tilde{J}_{q_1}(g_1(\theta) - q_1) + I(q_1) + \tilde{J}_{q_2}(\tau_{-k}g_2(\theta) - q_2) + I(q_2),
\]

where \( k \in \mathbb{Z} \) will be that of (ii) in Definition 3.1.22. In order for (3.161) to make sense, we need to know that in fact \( g_1(\theta) - q_1, \tau_{-k}g_2(\theta) - q_2 \in E \). But this follows from Lemma 1.4 of the chapter on mountain pass points by noting that \( g_1(\theta), q_1 \in \hat{E} \), both have the same asymptotics and \( I(g_1(\theta)) < \infty \). An analogous argument shows that \( \tau_{-k}g_2(\theta) - q_2 \in E \).

**Lemma 3.1.23.** Let \( h_1(\theta) := g_1(\theta) - q_1 \), and \( h_2(\theta) := \tau_{-k}g_2(\theta) - q_2 \). Then \( h_i \in C([0,1]^2, E) \) for \( i = 1, 2 \).
Proof. Suppose that $\theta_n \to \theta$ as $n \to \infty$ in $[0, 1]^2$. Then

\[
\|h_1(\theta_n) - h_1(\theta)\|_E^2 = \|g_1(\theta_n) - g_1(\theta)\|_E^2 \\
= \int_{-\infty}^{\infty} |g_1(\theta_n) - g_1(\theta)|^2 + |\nabla g_1(\theta_n) - \nabla g_1(\theta)|^2 \, dt \\
\leq \int_{\mathbb{R}} |g(\theta_n) - g(\theta)|^2 + |\nabla g(\theta_n) - \nabla g(\theta)|^2 \, dt \\
= \|g(\theta_n) - g(\theta)\|_E^2
\]

which goes to zero as $n \to \infty$, since $g \in C([0, 1]^2, E)$. A similar argument shows that $h_2 \in C([0, 1]^2, E)$.

Now, what points do the paths $h_1, h_2$ connect in $E$? We have

\[
h_1(0, \theta_2) = g_1(0, \theta_2) - q_1 \in B_d(0) \text{ for all } \theta_2 \in [0, 1].
\]

Next, $h_2(\theta_1, 0) = \tau_k g_2(\theta_1, 0) - q_2$, and thus

\[
\|h_2(\theta_1, 0)\|_E = \|\tau_k g_2(\theta_1, 0) - q_2\|_E = \|g_2(\theta_1, 0) - \tau_k q_2\| < d \text{ for all } \theta_1 \in [0, 1].
\]

Therefore, we have $h_1(0, \theta_2), h_2(\theta_1, 0) \in B_d(0)$ for all $\theta_1, \theta_2 \in [0, 1]$. Next, we have

\[
h_1(1, \theta_2) - (\tau_1 q_1 - q_1) = g_1(1, \theta_2) - q_1 - (\tau_1 q_1 - q_1) = g_1(1, \theta_2) - \tau_1 q_1
\]

and thus

\[
\|h_1(1, \theta_2) - (\tau_1 q_1 - q_1)\|_E = \|g_1(1, \theta_2) - \tau_1 q_1\|_E < d \text{ for all } \theta_2 \in [0, 1].
\]

Similarly,

\[
h_2(\theta_1, 1) - (\tau_1 q_2 - q_2) = \tau_k g_2(\theta_1, 1) - \tau_1 q_2
\]
and so

\[ \| h_2(\theta_1, 1) - (\tau_1 q_2 - q_2) \|_E = \| g_2(\theta_1, 1) - \tau_k \tau_1 q_2 \|_E < d \text{ for all } \theta_1 \in [0, 1]^2. \] (3.168)

Thus, for any choice of path \( f_1 \in C([0, 1], [0, 1]^2) \) such that \( f_1(0) \in \{0\} \times [0, 1] \) and \( f_1(1) \in \{1\} \times [0, 1] \), \( h_1 \circ f_1 \in \Gamma_{1,d} \). Similarly, for any path \( f_2 \in C([0, 1], [0, 1]^2) \) such that \( f_2(0) \in [0, 1] \times \{0\} \) and \( f_2(1) \in \{1\} \times [0, 1] \), \( h_2 \circ f_2 \in \Gamma_{2,d} \). Notice that we now have a method for taking a \( g \in \Gamma(d) \) and creating paths in \( \Gamma_{i,d} \).

**Lemma 3.1.24.** Let \( g \in \Gamma(d) \) for \( d \leq d_0 \). Then, there is a \( \tilde{\theta} \in [0, 1]^2 \) such that

\[ I(g(\tilde{\theta})) \geq \hat{c}_1 + I(q_1) + \hat{c}_2 + I(q_2) =: \bar{c}. \]

**Proof.** For every \( f_1 \in C([0, 1], [0, 1]^2) \) with \( f_1(0) \in \{0\} \times [0, 1] \) and \( f_1(1) \in \{1\} \times [0, 1] \), we have \( h_1 \circ f_1 \in \Gamma_{1,d} \) by Lemma 3.1.23, and so by Lemma 3.1.21, there is a \( \hat{\theta} \in [0, 1] \) such that

\[
\hat{c}_1 \leq J_{q_1}(h_1(f_1(\hat{\theta}))) = I(h_1(f_1(\hat{\theta}))) + q_1 - I(q_1) = I(g_1(f_1(\hat{\theta}))) - I(q_1). \] (3.169)

Hence

\[
\hat{c}_1 + I(q_1) \leq I(g_1(f_1(\hat{\theta}))). \] (3.170)

Since (3.170) is true for any \( f_1 \in C([0, 1], [0, 1]^2) \) with \( f_1(0) \in \{0\} \times [0, 1] \) and \( f_1(1) \in \{1\} \times [0, 1] \), we must have \( (g_1 \circ I)^{-1}(\{\hat{c}_1 + I(q_1)\}) \) separating \( \{0\} \times [0, 1] \) from \( \{1\} \times [0, 1] \) in \([0, 1]^2\). But then for any small \( \varepsilon > 0 \), there is a \( \delta_1 > 0 \) such that

\[ I(g_1(\theta)) \geq \hat{c}_1 + I(q_1) - \varepsilon \text{ for all } \theta \in N_{\delta_1}((g_1 \circ I)^{-1}(\{\hat{c}_1 + I(q_1)\})). \] Since \([0, 1]^2\) is compact, there are only finitely many components of \( N_{\delta_1}((g_1 \circ I)^{-1}(\{\hat{c}_1 + I(q_1)\}))\),
at least one of which separates \( \{0\} \times [0,1] \) from \( \{1\} \times [0,1] \). Let \( D_1 \) be such a component. Since \( D_1 \subset N_{\delta_1} \left( (g_1 \circ I)^{-1}(\{\hat{c}_1 + I(q_1)\}) \right) \), \( D_1 \) must be a path-connected set with

\[
D_1 \cap ([0,1] \times \{0\}) \neq \emptyset \\
D_1 \cap ([0,1] \times \{1\}) \neq \emptyset \tag{3.171}
\]

Thus, there is an \( f_2 \in C([0,1],[0,1]^2) \) such that \( f_2(0) \in [0,1] \times \{0\} \), \( f_2(1) \in [0,1] \times \{1\} \) and \( f_2([0,1]) \subset D_1 \). But by Lemma 3.1.23, \( f_2 \circ h_2 \in \Gamma_{2,d} \) and Lemma 3.1.21, there must be \( \hat{\theta}_2 \in [0,1] \) such that

\[
\dot{c}_2 \leq \tilde{J}_{q_2} \left( h_2(f_2(\hat{\theta}_2)) \right) = I(\tau_{-k}g_2(f_2(\hat{\theta}_2))) - I(q_2) \\
= I(g_2(f_2(\hat{\theta}_2))) - I(q_2). \tag{3.172}
\]

Since \( f_2(\hat{\theta}_2) \in D_1 \), we must have

\[
\hat{c}_1 + I(q_1) - \varepsilon \leq I(g_1(f_2(\hat{\theta}_2))) \tag{3.173}
\]

and so by (3.161), (3.172) and (3.173), we will have

\[
I(g(f_2(\hat{\theta}_2))) = I(g_1(f_2(\hat{\theta}_2))) + I(g_2(f_2(\hat{\theta}_2))) \\
\geq \hat{c}_1 + I(q_1) - \varepsilon + \dot{c}_2 + I(q_2) = \bar{c} - \varepsilon. \tag{3.174}
\]

Taking \( \theta_\varepsilon := f_2(\hat{\theta}_2) \), we will have a sequence of points \( \{\theta_\varepsilon\} \subset [0,1]^2 \) such that

\[
I(g(\theta_\varepsilon)) \geq \bar{c} - \varepsilon \tag{3.175}
\]

Since \([0,1]^2\) is compact, passing to a subsequence, we will have \( \theta_\varepsilon \rightarrow \bar{\theta} \) as \( \varepsilon \rightarrow 0 \). But then passing to the limit in (3.175) implies

\[
I(g(\bar{\theta})) \geq \bar{c}, \tag{3.176}
\]

which finishes the proof.
Moreover, we will want \( g \in \tau \) to have the property that when \( \hat{\theta}, \kappa > 0 \) is close to \( \text{A} \), and create a \( g \in \Gamma(d) \) with the property that \( I(g(\theta_1, \theta_2)) \) is close to \( \sum_{i=1,2} (\hat{J}_i(\gamma_i(\theta))) + I(q_i)) \).

Moreover, we will want \( g_1(\theta_1) - q_1 \) to be close to \( \gamma_1(\theta_1) \), and \( g_2(\theta_2) - \tau_k q_2 \) to be close to \( \tau_k \gamma_2(\theta_2) \). For this purpose, we proceed more generally. Consider any \( \gamma_i \in \Gamma_i \), and let \( \kappa > 0 \) be given. There is an \( M(\kappa) > 0 \) such that

\[
\| \gamma_i(\theta) \|_{W^1,2((-\infty, -M(\kappa))) \cup (M(\kappa), \infty))} < \kappa \tag{3.177}
\]

and

\[
\| \gamma_i(\theta) \|_{L^\infty((-\infty, -M(\kappa))) \cup (M(\kappa), \infty))} < \kappa \tag{3.178}
\]

uniformly in \( \theta_i \in [0, 1] \). Let \( \hat{\gamma}_i, \kappa(\theta) \) be defined as follows:

\[
\hat{\gamma}_i, \kappa(\theta)(t) := \begin{cases} 
0 & \text{for } t < -M(\kappa) - 1 \\
(t + M(\kappa) + 1)\gamma_i(\theta_i)(-M(\kappa)) & \text{for } t \in [-M(\kappa) - 1, -M(\kappa)] \\
\gamma_i(\theta_i)(t) & \text{for } -M(\kappa) < t < M(\kappa) \\
(M(\kappa) + 1 - t)\gamma_i(M(\kappa)) & \text{for } t \in [M(\kappa), M(\kappa) + 1] \\
0 & \text{for } t > M(\kappa) + 1
\end{cases}
\tag{3.179}
\]

Then \( \hat{\gamma}_i, \kappa \in C([0, 1], E) \) for \( i = 1, 2 \) and (writing \( M \) for \( M(\kappa) \)) we have

\[
\| \gamma_i(\theta_i) - \hat{\gamma}_i, \kappa(\theta_i) \|_{E}^2 \\
= \int_{-\infty}^{-M-1} (|\gamma_i(\theta_i)(t)|^2 + |\nabla \gamma_i(\theta_i)(t)|^2) \, dt + \int_{M+1}^{\infty} (|\gamma_i(\theta_i)(t)|^2 + |\nabla \gamma_i(\theta_i)(t)|^2) \, dt \\
+ \int_{-M-1}^{-M} (|\gamma_i(\theta_i)(t) - (t + M + 1)\gamma_i(\theta_i)(-M)|^2 + |\nabla \gamma_i(\theta_i)(t) - \gamma_i(\theta_i)(-M)|^2) \, dt \\
+ \int_{M+1}^{\infty} |\gamma_i(\theta_i)(t) - (M + 1 - t)\gamma_i(\theta_i)(M)|^2 + |\nabla \gamma_i(\theta_i)(t) - \gamma_i(\theta_i)(M)|^2 \, dt
\tag{3.180}
\]
Therefore,

\[
\|\gamma_i(\theta_i) - \hat{\gamma}_{i,\kappa}(\theta_i)\|^2_E \\
\leq \int_{-\infty}^{-M} \left( |\gamma_i(\theta_i)(t)|^2 + |\nabla \gamma_i(\theta_i)(t)|^2 \right) dt + \int_{M+1}^{\infty} \left( |\gamma_i(\theta_i)(t)|^2 + |\nabla \gamma_i(\theta_i)(t)|^2 \right) dt \\
+ 2 \int_{-M}^{-M-1} \left( |\gamma_i(\theta_i)(t)|^2 + |\nabla \gamma_i(\theta_i)(t)|^2 \right) \left( 1 + (t + M + 1)^2 \right) |\gamma_i(\theta_i)(-M)| dt \\
+ 2 \int_{M}^{M+1} \left( |\gamma_i(\theta_i)(t)|^2 + |\nabla \gamma_i(\theta_i)(t)|^2 \right) \left( 1 + (M + 1 - t)^2 \right) |\gamma_i(\theta_i)(M)|^2 dt \\
\leq 2\|\gamma_i(\theta_i)\|_{W^{1,2}([\omega(\lambda,\rho),\infty))}^2 + 4|\gamma_i(\theta_i)(-M)|^2 + 4|\gamma_i(\theta_i)(M)|^2 \\
\leq 2\kappa^2 + 4\kappa^2 + 4\kappa^2 < 16\kappa^2 \tag{3.181}
\]

Notice that (3.181) implies that

\[
\|\hat{\gamma}_{i,\kappa}(\theta_i) - \gamma_i(\theta_i)\| < 4\kappa. \tag{3.182}
\]

Hence \(\hat{\gamma}_{i,\kappa}(0) \in B_{3\kappa}(0)\) and \(\hat{\gamma}_{i,\kappa}(1) \in B_{3\kappa}(\tau_1 q_i - q_i)\). Moreover, we know that for \(t < -M(\kappa) - 1\) and \(t > M(\kappa) + 1\), we will have

\[
\hat{\gamma}_{i,\kappa}(\theta_i)(t) = 0. \tag{3.183}
\]

Now, for any \(\rho < \rho^*(2r)\) sufficiently small that \(\omega(A_1, \rho) + 1 > M(\kappa) + 1\) and \(\alpha(A_2, \rho) - 1 < -M(\kappa) - 1\), (3.183) implies that

\[
q_{1A_1,\rho}(t) + \hat{\gamma}_{1,\kappa}(\theta_1)(t) = \xi \text{ for } t > \omega(A_1, \rho) + 1 \text{ and }
\]

\[
q_{2A_2,\rho}(t) + \hat{\gamma}_{2,\kappa}(\theta_2)(t) = \xi \text{ for } t < \alpha(A_2, \rho) - 1 \tag{3.184}
\]

Then, for any \(\rho < \rho^*(2r)\) sufficiently small that \(\omega(A_1, \rho) + 1 > M(\kappa) + 1\) and \(\alpha(A_2, \rho) - 1 < -M(\kappa) - 1\) and for any \(n \in \mathbb{N}\) sufficiently large that \(\omega(A_1, \rho) + 1 <
Remark 3.1.26. Notice that to define $\alpha(A_1, \rho) + n - 1$, let us define

$$g_{\kappa, \rho, n}(\theta_1, \theta_2)(t) := \begin{cases} \left(q_{1, A_1, \rho} + \check{\gamma}_{1, \kappa}(\theta_1)\right)(t) & \text{for } t < \omega(A_1, \rho) + 1 \\ \xi & \text{for } t \in [\omega(A_1, \rho) + 1, \alpha(A_2, \rho) + n - 1] \\ \tau_n \left(q_{2, A_2, \rho} + \check{\gamma}_{2, \kappa}(\theta_2)\right)(t) & \text{for } t > \alpha(q_2, \rho) + n - 1 \end{cases}$$

(3.185)

Notice that we must have $g_{\kappa, \rho, n}(\theta_1, \theta_2) \in E$ for all $(\theta_1, \theta_2) \in [0, 1]^2$. To see this, all that we need to note is that $(q_{i, A_i, \rho} + \check{\gamma}_{i, \kappa}(\theta_i)) \in \hat{E}$ for $i = 1, 2$ and $g_{\kappa, \rho, n}(\theta_1, \theta_2)$ has the correct asymptotics to be an element of $E$. Next, suppose that $(\theta_1, n, \theta_2, n) \to (\theta_1, \theta_2)$ as $n \to \infty$. Then, we will have

$$\|g_{\kappa, \rho, n}(\theta_1, n, \theta_2, n) - g_{\kappa, \rho, n}(\theta_1, \theta_2)\| \leq \|\check{\gamma}_{1, \kappa}(\theta_1, n) - \check{\gamma}_{1, \kappa}(\theta_1)\| W^{1,2}(\alpha(A_1, \rho) + 1)$$

$$+ \|\tau_n \left(\check{\gamma}_{2, \kappa}(\theta_2, n) - \check{\gamma}_{2, \kappa}(\theta_2)\right)\| W^{1,2}(\alpha(A_2, \rho) + n + 1)$$

(3.186)

$$\leq \|\check{\gamma}_{1, \kappa}(\theta_1, n) - \check{\gamma}_{1, \kappa}(\theta_1)\| + \|\check{\gamma}_{2, \kappa}(\theta_2, n) - \check{\gamma}_{2, \kappa}(\theta_2)\| \to 0$$

as $n \to \infty$, since each $\check{\gamma}_{i, \kappa} \in C([0, 1], E)$. Therefore, $g_{\kappa, \rho, n} \in C([0, 1]^2, E)$.

Lemma 3.1.25. If $r < d$ and $\kappa \leq \frac{d}{8}$, then $g_{\kappa, \rho, n}(\theta) \in \Gamma(d)$ for all $\rho \leq \rho^*(2r)$.

Remark 3.1.26. Notice that to define $g_{\kappa, \rho, n}(\theta_1, \theta_2)$, we first pick $\kappa$. Then, we take $\rho$ sufficiently small that $\omega(A_1, \rho) + 1 > M(\kappa) + 1$ and $\alpha(A_2, \rho) - 1 < -M(\kappa) - 1$. Finally, we take any $n \in \mathbb{N}$ sufficiently large that $\omega(A_1, \rho) + 1 < \alpha(A_2, \rho) + n - 1$.

Proof. Notice that we may take $a_1 = \omega(q_1, \rho) + 1$ and $a_2 = \alpha(q_2, \rho) + n - 1$ and satisfy (i) of Definition 3.1.22. Next, notice that if $g_1, g_2$ are defined as in (3.158) and (3.159), we will have

$$g_1(\theta_1) \equiv q_{1, A_1, \rho} + \check{\gamma}_{1, \kappa}(\theta_1) \text{ and } g_2(\theta_2) \equiv \tau_n \left(q_{2, A_2, \rho} + \check{\gamma}_{2, \kappa}(\theta_2)\right).$$

(3.187)
Notice that by Lemma 3.1.17, (3.182) and (3.187),

\[
\|g_1(\theta_1) - q_1 - \gamma_1(\theta_1)\|_E \leq \|q_{A_1,\rho} - q_1\|_E + \|\hat{\gamma}_{1,\kappa}(\theta_1) - \gamma_1(\theta_1)\|_E \\
\leq \frac{r}{64} + 4\kappa.
\]  

(3.188)

since \(\rho < \rho^*(2r)\). Picking \(\kappa < d/8\) and \(r < d\), (3.188) implies that \(\|g_1(0) - q_1\|_E < d\), since \(\gamma_1(0) = 0\), and \(\|g_1(1) - \tau_1 q_1\|_E < d\), since \(\gamma_1(1) = \tau_1 q_1 - q_1\). Similarly, Lemma 3.1.17, (3.182) and (3.187) implies that

\[
\|g_2(\theta_2) - \tau_n q_2 - \tau_n \gamma_2(\theta_2)\|_E = \|\tau_n (q_{2,\rho} + \hat{\gamma}_{2,\kappa}(\theta_2)) - \tau_n q_2 - \tau_n (\gamma_2(\theta_2))\|_E \\
\leq \|q_2 - q_2\|_E + \|\hat{\gamma}_{2,\kappa}(\theta_2) - \gamma_2(\theta_2)\|_E \\
\leq \frac{r}{64} + 4\kappa < d.
\]  

(3.189)

Arguing as above, we see that \(\|g_2(0) - \tau_n q_2\|_E < d\), since \(\gamma_2(0) = 0\). Similarly, since \(\gamma_2(1) = \tau_1 q_2 - q_2\), \(\|g_2(1) - \tau_n q_2\|_E < d\), and thus we will have (ii) of Definition 3.1.22 satisfied, and so \(g_{\kappa,\rho,n} \in \Gamma(d)\).

Notice that by (3.160), we have

\[
I(g_{\kappa,\rho,n}(\theta_1, \theta_2)) = I(\hat{\gamma}_{1,\kappa} + q_{1,A_1,\rho}) + I(\tau_n(\hat{\gamma}_{2,\kappa} + q_{A_2,\rho})) \\
= I(\hat{\gamma}_{1,\kappa} + q_{1,A_1,\rho}) + I(\hat{\gamma}_{2,\kappa} + q_{A_2,\rho}).
\]  

(3.190)

Thus, if \(g_{\kappa,\rho,n}(\theta_1, \theta_2)\) is as defined in (3.185), then (3.190) implies that

\[
I(g_{\kappa,\rho,n}(\theta_1, \theta_2)) = I(\hat{\gamma}_{1,\kappa}(\theta_1) + q_{1,A_1,\rho}) + I(\hat{\gamma}_{2,\kappa}(\theta_2) + q_{A_2,\rho}) \\
= \sum_{i=1,2} \left( \tilde{J}_{q_i}(\hat{\gamma}_{i,\kappa}(\theta_i) + q_{i,A_i,\rho} - q_i) + I(q_i) \right) \\
= \sum_{i=1,2} \left( \tilde{J}_{q_i}(\hat{\gamma}_{i,\kappa,\rho}(\theta_i)) + I(q_i) \right),
\]  

(3.191)
where
\[
\tilde{\gamma}_{i,\kappa,\rho}(\theta_i) := \hat{\gamma}_{i,\kappa}(\theta_i) + q_{i,A_i,\rho} - q_i.
\] (3.192)

Notice that we have \(\tilde{\gamma}_{i,\kappa,\rho}(\theta_i) \in C([0,1], E)\), and
\[
\|\tilde{\gamma}_{i,\kappa,\rho}(\theta_i) - \gamma_i(\theta_i)\|_E \leq \|\hat{\gamma}_{i,\kappa}(\theta_i) - \gamma_i(\theta_i)\|_E + \|q_{i,A_i,\rho} - q_i\|_E
\] (3.193)
\[
\leq 4\kappa + \frac{r}{64} \quad \text{(3.194)}
\]
since \(\rho < \rho^*(2r)\).

We would next like to know the relation between \(c(d)\) and \(\hat{c}_1, \hat{c}_2\), which we can do using Lemma 3.1.21.

**Proposition 3.1.27.** If \(d \leq d_0\), where \(d_0\) is from Lemma 3.1.21, then
\[
c(d) = \bar{c}.
\]
where \(\bar{c}\) is defined in Lemma 3.1.24.

**Proof.** By Lemma 3.1.24, we must have \(\bar{c} \leq c(d)\).

Next, let \(\varepsilon > 0\) be given. By definition of \(\hat{c}_i\), there are paths \(\gamma_i \in \Gamma_i\) such that
\[
\max_{\theta_i \in [0,1]} J_{q_i}(\gamma_i(\theta_i)) < \hat{c}_i + \varepsilon/4.
\] By taking \(\kappa\) and \(\rho\) to be very small, we will be able to have \(\tilde{\gamma}_{i,\kappa,\rho}(\theta_i)\) uniformly close enough to \(\gamma_i(\theta_i)\) that
\[
\max_{\theta_i \in [0,1]} J_{q_i}(\tilde{\gamma}_{i,\kappa,\rho}(\theta_i)) \leq \hat{c}_i + \varepsilon/2. \quad \text{(3.195)}
\]

Next, let \(g\) be the element of \(\Gamma(d)\) associated to \(\gamma_i\), as in (3.185). Suppose now that \((\theta_{1, n}, \theta_{2, n}) \subset [0,1]^2\) is a sequence with \(I(g(\theta_{1, n}, \theta_{2, n})) \to \max_{\theta \in [0,1]^2} I(g(\theta))\) as \(n \to \infty\). Because \([0,1]\) is compact, after passing to a subsequence, we will have
\( (\theta_{1,n}, \theta_{2,n}) \to (\theta_1, \theta_2) \) as \( n \to \infty \). Thus, using (3.191), we have

\[
I(g(\theta_{1,n}, \theta_{2,n})) = \sum_{i=1,2} \left( \tilde{J}_{q_i}(\tilde{\gamma}_{i,n,\rho}(\theta_{i,n})) + I(q_i) \right).
\] (3.196)

Passing to the limit as \( n \to \infty \), (3.196) becomes

\[
\max_{\theta \in [0,1]^2} I(g(\theta)) = \sum_{i=1,2} \left( \tilde{J}_{q_i}(\tilde{\gamma}_{i,n,\rho}(\theta)) + I(q_i) \right) \leq \sum_{i=1,2} (\hat{c}_i + I(q_i) + \varepsilon/2)
\] (3.197)

and so \( \max_{\theta \in [0,1]^2} I(g(\theta)) < \bar{c} + \varepsilon \). Thus, \( c(d) < \bar{c} + \varepsilon \). Since this is true for every \( \varepsilon > 0 \), we must have \( c(d) \leq \bar{c} \), and the proof is complete.

\( \square \)

### 3.2 Two-Bumps

Now, we can finally turn to the existence of multi-bump solutions of (HS) that are close to the chain \((u_1 + q_1, u_2 + q_2)\), where \( u_i \in \mathcal{K}(\tilde{J}_{q_i})_{\hat{c}_i + \varepsilon}. \) Recall that there is a constant \( C \) such that

\[
\|u\|_{L^\infty} \leq C\|u\|.
\] (3.198)

The precise theorem that we wish to prove is:

**Theorem 3.2.1.** If

\[
r < \min \left\{ \frac{\mu}{4}, \frac{d_0}{4C}, \frac{\eta}{4} \right\} \min \left\{ \frac{1}{2}, \beta_2 \right\} \max \left\{ \frac{1}{2}, \beta_1 \right\}
\]

where \( C \) is from (3.198), there is a \( 0 < \tilde{\rho}(r) \leq \rho^*(2r) \) such that if \( \rho < \tilde{\rho}(r) \), then \( \mathcal{N}(2r, \rho) \) is finite, where \( \mathcal{N}(r, \rho) \) is defined by (3.97).

We prove Theorem 3.2.1 by contradiction. We shall construct an element \( g \in \Gamma(d_0) \) such that \( \max_{\theta \in [0,1]^2} I(g(\theta)) < \bar{c} \), contradicting Proposition 3.1.27.
Proof. Let
\[ \varepsilon := \min \left\{ \frac{r\delta(A_1, A_2, r)}{32}, \frac{\mu \cdot d_0}{4}, \hat{c}_1, \hat{c}_2 \right\} \] (3.199)
where \( \delta(A_1, A_2, r) \) is from Proposition 3.1.20. By Proposition 3.1.11, there exist \( \gamma_i \in \Gamma_i \) such that
\[ (1) \max_{\theta_i \in [0, 1]} \tilde{J}_{\eta_i}(\gamma_i(\theta_i)) < \frac{\tilde{c}_i + \varepsilon}{4} \] (3.200)
\[ (2) \text{If} \quad \tilde{J}_{\eta_i}(\gamma_i(\theta_i)) > \hat{c}_i - \varepsilon, \text{then} \quad \gamma_i(\theta_i) \in N_{r/16}(A_i). \]
Thus, \( \gamma_i \) depends on \( \varepsilon \) and \( r \).

Lemma 3.2.2. There is a \( \hat{\kappa}(r, \varepsilon) > 0 \) and \( \hat{\rho}(r, \varepsilon) > 0 \) such that whenever \( \kappa < \max\{d_0, \hat{\kappa}(r, \varepsilon)\} \) and \( \rho < \min\{\rho^*(2r), \hat{\rho}(r, \varepsilon)\} \) then
\[ (i) \quad \|\gamma_i(\theta) - (\hat{\gamma}_{i,\kappa}(\theta_i) + q_{i, A_i, \rho} - q_i)\| < \frac{r}{16} \]
\[ (ii) \quad \|\tilde{J}_{\eta_i}(\gamma_i(\theta_i)) - \tilde{J}_{\eta_i}(\hat{\gamma}_{i,\kappa}(\theta_i) + q_{i, A_i, \rho} - q_i)\| < \frac{\varepsilon}{12}. \]

Proof. To see this, let \( \kappa_1(r) \) be so small that if \( \kappa < \kappa_1(r) \) then
\[ \|\gamma_i(\theta_i) - \hat{\gamma}_{i,\kappa}(\theta_i)\| < \frac{r}{32}, \] (3.201)
and let \( \rho_1 < \rho^*(2r) \) be so small that if \( \rho < \rho_1 \), then
\[ \|q_{i, A_i, \rho} - q_i\| < \frac{r}{32}. \] (3.202)
Thus, if \( \kappa < \kappa_1 \) and \( \rho < \rho_1 \), (3.201) and (3.202) imply (i) of Lemma 3.2.2. Next, let \( \bar{K}(\gamma_1, \gamma_2) = \bar{K}(\varepsilon, r) \) be so large that
\[ \|I'(\gamma_i(\theta_i))\| \leq \bar{K}(r, \varepsilon) \quad \text{for} \quad i = 1, 2, \theta_i \in [0, 1]. \] (3.203)
Since $I$ is Lipschitz, we will then know that for any $u \in N_{r/16}(\gamma_i([0,1]))$

$$\|I'(u)\| \leq \|I'(u) - I'(\gamma_i(\theta_i))\| + \|I'(\gamma_i(\theta_i))\|$$

$$\leq K \frac{r}{16} + \bar{K}(r, \varepsilon). \tag{3.204}$$

Thus,

$$\tilde{J}_{q_i}(\hat{\gamma}_{i,\kappa}(\theta_i) + q_{iA_i,\rho} - q_i) - \tilde{J}_{q_i}(\gamma_i(\theta_i))$$

$$= \int_0^1 \tilde{J}_{q_i}'((s(\hat{\gamma}_{i,\kappa}(\theta_i) + q_{iA_i,\rho} - q_i) + (1 - s)\gamma_i(\theta_i))(\hat{\gamma}_{i,\kappa}(\theta_i) + q_{iA_i,\rho} - q_i - \gamma_i(\theta_i))\)ds. \tag{3.205}$$

Now, if $\delta < \delta_1(r)$ and $\rho < \rho_1(r)$, then by (i) of Lemma 3.2.2, we will have

$$\left|\tilde{J}_{q_i}(\hat{\gamma}_{i,\kappa}(\theta_i) + q_{iA_i,\rho} - q_i) - \tilde{J}_{q_i}(\gamma_i(\theta_i))\right|$$

$$\leq \left(\int_0^1 \|\tilde{J}_{q_i}'(\gamma(\theta_i) + s(\hat{\gamma}_{i,\kappa}(\theta_i) - \gamma_i(\theta_i) + q_{iA_i,\rho} - q_i))\|ds\right)$$

$$\times \|\gamma_i(\theta_i) - \hat{\gamma}_{i,\kappa}(\theta_i) + q_{iA_i,\rho} - q_i\|$$

$$\leq \left(K \frac{r}{16} + \bar{K}(r, \varepsilon)\right)(\|\hat{\gamma}_{i,\kappa}(\theta_i) - \gamma_i(\theta_i)\| + \|q_{iA_i,\rho} - q_i\|). \tag{3.206}$$

Now, by (3.182), there is a $\kappa_2(r, \varepsilon) \leq \kappa_1(r)$ such that if $\kappa < \kappa_2(r, \varepsilon)$, then

$$\|\gamma_{i,\kappa}(\theta_i) - \gamma(\theta_i)\| \leq \frac{\varepsilon}{24 \left(K \frac{r}{16} + \bar{K}(r, \varepsilon)\right)} \text{ for all } \theta_i \in [0, 1]. \tag{3.207}$$

Similarly, there is a $\rho_2(r, \varepsilon) \leq \rho_1(r)$ such that if $\rho \leq \rho_2(r, \varepsilon)$, then

$$\|q_{iA_i,\rho} - q_i\| \leq \frac{\varepsilon}{24 \left(K \frac{r}{16} + \bar{K}(r, \varepsilon)\right)}. \tag{3.208}$$

Thus, if $\kappa \leq \kappa_2(r, \varepsilon) =: \hat{\kappa}$ and $\rho < \rho_2(r, \varepsilon) =: \hat{\rho}$, then combining (3.206)-(3.208), we will have

$$\left|\tilde{J}_{q_i}(\hat{\gamma}_{i,\kappa}(\theta_i) + q_{iA_i,\rho} - q_i) - \tilde{J}_{q_i}(\gamma_i(\theta_i))\right| < \frac{\varepsilon}{12}, \tag{3.209}$$

which gives us (ii) of (3.2.2). Moreover, since $\kappa_2(r, \varepsilon) \leq \kappa_1(r)$ and $\rho_2(r, \varepsilon) \leq \rho_1(r)$, we will also have (i) of (3.2.2) satisfied, which finishes the proof of Lemma 3.2.2. □
Now, let \( \kappa < \kappa_2(r, \varepsilon) \) be fixed. Thus, there is a \( \rho_3 \) depending on \( \kappa \) with \( \rho_3 \leq \rho_2(r, \varepsilon) \) such that if \( \rho \leq \rho_3 \), then \( \omega(A_1, \rho) + 1 > M(\kappa) + 1 \) and \( \alpha(A_2, \rho) - 1 < -M(\kappa) - 1 \). Let \( \tilde{\rho} := \rho_3 \), and fix \( \rho < \tilde{\rho} \). Since \( N(2r, \rho) \) is assumed to be infinite, there are an infinite number of \( n \in N(2r, \rho) \) such that \( \omega(A_1, \rho) + 1 < \alpha(A_2, \rho) + n - 1 \). For any such \( n \), let

\[
g_{\kappa, \rho, n}(\theta_1, \theta_2)(t) := \begin{cases} (q_{1A_1, \rho} + \hat{\gamma}_{1, \kappa}(\theta_1))(t) & \text{for } t < \omega(A_1, \rho) + 1 \\ \xi & \text{for } t \in [\omega(A_1, \rho) + 1, \alpha(A_2, \rho) + n - 1] \\ \tau_n \left( q_{2A_2, \rho} + \hat{\gamma}_{2, \kappa}(\theta_2) \right)(t) & \text{for } t > \alpha(q_2, \rho) + n - 1 \end{cases}
\]  

(3.210)

as in (3.185). Notice that if we take \( \kappa < \frac{d_0}{8} \) and \( r < d_0 \), we will have

\[
g_{\kappa, \rho, n}(\theta_1, \theta_2) \in \Gamma(d_0)
\]

(3.211)

by Lemma 3.1.25. We have

\[
(g_{\kappa, \rho, n})_1(\theta_1, \theta_2)(t) := \begin{cases} (q_{1A_1, \rho} + \hat{\gamma}_{1, \kappa}(\theta_1))(t) & \text{for } t < \omega(A_1, \rho) + 1 \\ \xi & \text{for } t \geq \omega(A_1, \rho) + 1 \end{cases}
\]

(3.212)

\[
= q_{1A_1, \rho}(t) + \hat{\gamma}_{1, \kappa}(\theta_1)(t)
\]

and

\[
(g_{\kappa, \rho, n})_2(\theta_1, \theta_2)(t) := \begin{cases} \xi & \text{for } t \leq \alpha(A_2, \rho) + n - 1 \\ \tau_n \left( q_{2A_2, \rho} + \hat{\gamma}_{2, \kappa}(\theta_2) \right)(t) & \text{for } t > \alpha(A_2, \rho) + n - 1 \end{cases}
\]

(3.213)

\[
= \tau_n(q_{2A_2, \rho}(t) + \hat{\gamma}_{2, \kappa}(\theta_2)(t))
\]
Since $g_{\kappa,\rho,n} \in \Gamma(d_0)$, (3.212) and (3.213) imply that

$$\|q_{1A_1,\rho} + \hat{\gamma}_{1,\kappa}(0) - q_1\| < d_0$$

$$\|q_{1A_1,\rho} + \hat{\gamma}_{1,\kappa}(1) - \tau_1 q_1\| < d_0$$ (3.214)

$$\|\tau_n(q_{2A_2,\rho} + \hat{\gamma}_{2,\kappa}(0)) - \tau_n q_2\| < d_0$$

$$\|\tau_n(q_{2A_2,\rho} + \hat{\gamma}_{2,\kappa}(1)) - \tau_n \tau_1 q_2\| < d_0$$

Moreover, since $\kappa < \kappa_2$ and $\rho < \rho_2$, by Lemma 3.2.2, we will know that $\hat{\gamma}_{i,\kappa}(\theta_i) + q_{iA_i,\rho} - q_i$ has the following properties:

$$\left| \tilde{J}_{q_i}(\hat{\gamma}_{i,\kappa}(\theta_i) + q_{iA_i,\rho} - q_i) - \tilde{J}_{q_i}(\gamma_i(\theta_i)) \right| < \frac{\varepsilon}{12}$$ (3.215)

$$\|\hat{\gamma}_{i,\kappa}(\theta_i) + q_{iA_i,\rho} - q_i - \gamma_i(\theta_i)\| < \frac{r}{16}$$

We have

$$I(g_{\kappa,\rho,n}(\theta_1, \theta_2)) = I(\hat{\gamma}_{1,\kappa}(\theta_1) + q_{1A_1,\rho}) + I(\hat{\gamma}_{2,\kappa}(\theta_2) + q_{2A_2,\rho})$$ (3.216)

$$= \sum_{i=1,2} \left( \tilde{J}_{q_i}(\hat{\gamma}_{i,\kappa} + q_{iA_i,\rho} - q_i) + I(q_i) \right)$$

Notice that (3.216) is independent of $n$.

**Lemma 3.2.3.** If

$$I(g_{\kappa,\rho,n}(\theta_1, \theta_2)) > \bar{c} - \frac{\varepsilon}{3}$$

then

$$\tilde{J}_{q_i}(\hat{\gamma}_{i,\kappa}(\theta_i) + q_{iA_i,\rho} - q_i) > \hat{c}_i - \frac{2\varepsilon}{3}$$

for both $i = 1, 2$.

**Proof.** If this is false, then for example, we will have

$$\tilde{J}_{q_1}(\hat{\gamma}_{1,\kappa}(\theta_1) + q_{1A_1,\rho} - q_1) \leq \hat{c}_1 - \frac{2\varepsilon}{3}. \quad (3.217)$$
But then by (3.216), we will have

\[ \bar{c} - \frac{\varepsilon}{3} < I(g_{\kappa,\rho,n}(\theta_1, \theta_2)) \]

\[ = \sum_{i=1,2} \left( \tilde{J}_{q_i}(\hat{\gamma}_i, \kappa + q_i A_i, \rho - q_i) + I(q_i) \right). \]  

(3.218)

But, combining (3.217) and (3.218), we have

\[ \bar{c} - \frac{\varepsilon}{3} \leq \hat{c}_1 + I(q_1) - \frac{2\varepsilon}{3} + \left( \tilde{J}_{q_2}(\hat{\gamma}_2, \kappa + q_2 A_2, \rho - q_2) + I(q_2) \right). \]

(3.219)

Now, by (1) of (3.200) and (3.215), we know that

\[ \tilde{J}_{q_2}(\hat{\gamma}_2, \kappa + q_2 A_2, \rho - q_2) < \hat{c}_2 + \frac{\varepsilon}{4} + \frac{\varepsilon}{12} = \hat{c}_2 + \frac{\varepsilon}{3}. \]  

(3.220)

Combining (3.219) and (3.220), we have

\[ \bar{c} - \frac{\varepsilon}{3} \leq \hat{c}_1 + I(q_1) - \frac{2\varepsilon}{3} + \hat{c}_2 + I(q_2) + \frac{\varepsilon}{3} = \bar{c} - \frac{\varepsilon}{3}, \]  

(3.221)

which is impossible, completing the proof of Lemma 3.2.3.

\[ \square \]

**Lemma 3.2.4.** If

\[ I(g_{\kappa,\rho,n}(\theta_1, \theta_2)) > \bar{c} - \frac{\varepsilon}{3}, \]

then

\[ g_{\kappa,\rho,n}(\theta_1, \theta_2) \in N_{\frac{\varepsilon}{3}}(\mathcal{M}(A_1, A_2, \rho, n)). \]

**Proof.** By Lemma 3.2.3, if \( I(g_{\kappa,\rho,n}(\theta_1, \theta_2)) > \bar{c} - \frac{\varepsilon}{3} \), then

\[ \tilde{J}_{q_i}(\hat{\gamma}_i, \kappa + q_i A_i, \rho - q_i) > \frac{2\varepsilon}{3} \]

for \( i = 1, 2 \). By (3.200) and (3.215), (3.222) implies that

\[ \tilde{J}_{q_i}(\gamma_i(\theta_i)) > \frac{2\varepsilon}{3} - \frac{\varepsilon}{12} > \hat{c}_i - \varepsilon, \]  

(3.223)
and so by (2) of (3.200)

$$\gamma_i(\theta_i) \in N_{\frac{r}{10}}(A_i)$$  \hspace{1cm} (3.224)

Applying (3.215) to (3.224), we see that we must have

$$\hat{\gamma}_{i,\kappa}(\theta_i) + q_{iA_i,\rho} - q_i \in N_{\frac{r}{8}}(A_i)$$ \hspace{1cm} (3.225)

for $i = 1, 2$. Thus, there exist $u_i \in A_i$ such that

$$\|u_i - (\hat{\gamma}_{i,\kappa}(\theta_i) + q_{iA_i,\rho} - q_i)\| < \frac{r}{8}.$$ \hspace{1cm} (3.226)

We now show

$$\|g_{\kappa,\rho,n}(\theta_1, \theta_2) - Q((u_1, u_2), \rho, n)\| \leq \frac{r}{2},$$ \hspace{1cm} (3.227)

which will prove the Lemma. By Definition 3.1.18, we have

$$Q((u_1, u_2), \rho, k)(t) := \begin{cases} (u_1 + q_1)_{A_1,\rho}(t) & \text{for } t \leq \omega(A_1, \rho) + 1 \\ \xi & \text{for } t \in [\omega(A_1, \rho) + 1, \alpha(A_2, \rho) + n - 1] \\ \tau_n(u_2 + q_2)_{A_2,\rho}(t) & \text{for } t > \alpha(A_2, \rho) + n - 1 \end{cases}$$

and by (3.185), we have

$$g_{\kappa,\rho,n}(\theta_1, \theta_2)(t) := \begin{cases} (q_{1A_1,\rho} + \hat{\gamma}_{1,\kappa}(\theta_1))(t) & \text{for } t < \omega(A_1, \rho) + 1 \\ \xi & \text{for } t \in [\omega(A_1, \rho) + 1, \alpha(A_2, \rho) + n - 1] \\ \tau_n(q_{2A_2,\rho} + \hat{\gamma}_{2,\kappa}(\theta_2))(t) & \text{for } t > \alpha(q_2, \rho) + n - 1 \end{cases}$$

Together, these imply that

$$\|g_{\kappa,\rho,n} - Q((u_1, u_2), \rho, n)\|^2 = \|(u_1 + q_1)_{A_1,\rho} - (\hat{\gamma}_{1,\kappa}(\theta_1) + q_{1A_1,\rho})\|_{W^{1,2}(-\infty, \omega(A_1, \rho) + 1)}^2$$

$$+ \|\tau_n(u_2 + q_2)_{A_2,\rho} - \tau_n(\hat{\gamma}_{2,\kappa}(\theta_2) + q_{2A_2,\rho})\|_{W^{1,2}(\alpha(A_2, \rho) + n - 1, \infty)}^2$$

$$= \|(u_1 + q_1)_{A_1,\rho} - (\hat{\gamma}_{1,\kappa}(\theta_1) + q_{1A_1,\rho})\|^2$$

$$+ \|(u_2 + q_2)_{A_2,\rho} - (\hat{\gamma}_{2,\kappa}(\theta_2) + q_{2A_2,\rho})\|^2. \hspace{1cm} (3.228)$$
Because $\rho < \rho^*(2r)$ and (3.226), we must have

$$
\|(u_1 + q_1)_{A_1,\rho} - (\hat{\gamma}_{1,\kappa} (\theta_1) + q_{1A_1,\rho})\| \leq \|(u_1 + q_1)_{A_1,\rho} - (u_1 + q_1)\|
$$

$$
+ \|(u_1 + q_1) - (\hat{\gamma}_{1,\kappa} (\theta_1) + q_{1A_1,\rho})\|
$$

$$
\leq \frac{r}{64} + \|u_1 - (\hat{\gamma}_{1,\kappa} (\theta_1) + q_{1A_1,\rho} - q_1)\|
$$

(3.229)

$$
\leq \frac{r}{64} + \frac{r}{8} < \frac{r}{4}.
$$

Similarly, we must have

$$
\|(u_2 + q_2)_{A_2,\rho} - (\hat{\gamma}_{2,\kappa} (\theta_2) + q_{2A_2,\rho})\| < \frac{r}{4}.
$$

(3.230)

Since $u_i \in A_i$, (3.228) - (3.230) show that (3.227) holds, and Lemma 3.2.4 is proved. \qed

Next, we need to show that for any $\theta_1, \theta_2 \in [0, 1]$, $I(g_{\kappa,\rho,n}(0, \theta_2))$, $I(g_{\kappa,\rho,n}(1, \theta_2))$, $I(g_{\kappa,\rho,n}(\theta_1, 0))$ and $I(g_{\kappa,\rho,n}(\theta_1, 1))$ are each smaller than $\bar{c} - \frac{\varepsilon}{3}$. By (3.200), (3.215) and (3.216), we know that

$$
I(g_{\kappa,\rho,n}(0, \theta_2)) = I(\hat{\gamma}_{1,\kappa}(0) + q_{1A_1,\rho}) + I(\hat{\gamma}_{2,\kappa}(\theta_2) + q_{2A_2,\rho})
$$

$$
= \tilde{J}_{q_1} (\hat{\gamma}_{1,\kappa}(0) + q_{1A_1,\rho} - q_1) + I(q_1) + \tilde{J}_{q_2} (\hat{\gamma}_{2,\kappa}(\theta_2) + q_{2A_2,\rho} - q_2)
$$

(3.231)

$$
\leq \tilde{J}_{q_1} (\gamma_1(0)) + \frac{\varepsilon}{12} + I(q_1) + \hat{c}_2 + \frac{\varepsilon}{3} + I(q_2)
$$

But $\tilde{J}_{q_1}(\gamma_1(0)) = 0$ since $\gamma_1(0) = 0$. Thus, (3.231) implies that

$$
I(g_{\kappa,\rho,n}(0, \theta_2)) \leq \frac{5\varepsilon}{12} + \hat{c}_2 + I(q_2) + I(q_1).
$$

(3.232)
By (3.199), we know that $\varepsilon \leq \hat{c}_1$, and so
\begin{equation}
I(g_{\kappa,\rho,n}(0, \theta_2)) \leq \frac{5\hat{c}_1}{12} + I(q_1) + \hat{c}_2 + I(q_2) = \bar{c} - \frac{7\hat{c}_1}{12}.
\end{equation}

Since $\varepsilon \leq \hat{c}_1$, we know that $\bar{c} - \frac{7\hat{c}_1}{12} \leq \bar{c} - \frac{7\varepsilon}{12} < \bar{c} - \frac{\varepsilon}{3}$, and we have a bound
\begin{equation}
I(g_{\kappa,\rho,n}(0, \theta_2)) < \bar{c} - \frac{\varepsilon}{3}.
\end{equation}

A similar argument provides (3.234) for $g_{\kappa,\rho,n}(1, \theta_2), g_{\kappa,\rho,n}(\theta_1, 0)$ and $g_{\kappa,\rho,n}(\theta_1, 1)$.

Next, we deform $g_{\kappa,\rho,n}(\theta_1, \theta_2)$ to $G(\theta_1, \theta_2) \in C([0, 1]^2, E)$ such that
\begin{equation}
\max_{(\theta_1, \theta_2) \in [0, 1]^2} I(G'(\theta_1, \theta_2)) < \bar{c} - \frac{\varepsilon}{3}.
\end{equation}

To do this, we must rely on the fact that $n \in \mathcal{N}(2r, \rho)$, and so we have a bound away from zero on the size of $I'(z)$ for $z \in N_{2r}(\mathcal{M}(A_1, A_2, \rho, n))$. Indeed, by Proposition 3.1.20, we have a bound that depends only on $A_1, A_2$ and $r$ such that $I'(z)$ is bounded away from zero in an “annular” neighborhood of $\mathcal{M}(A_1, A_2, \rho, n)$. Thus, we will be able to control how much we deform $g_{\kappa,\rho,n}(\theta_1, \theta_2)$. By making sure that we do not deform too far, we will know that on the interval $[\omega(A_1, \rho) + 1, \alpha(A_2, \rho) + n - 1]$, $G(\theta_1, \theta_1)(t)$ will be close to $\xi$. However, we cannot make sure that $G \in \Gamma(d_0)$. Instead, we will have to modify $G$ to get an element of $\Gamma(d_0)$.

Let
\begin{equation}
f(x) := \frac{\text{dist} (x, (I)_{\bar{c} - \varepsilon} \cup (I)_{\bar{c} + \varepsilon})}{\text{dist} (x, (I)_{\bar{c} - \varepsilon} \cup (I)_{\bar{c} + \varepsilon}) + \text{dist} (x, (I)_{\bar{c} - \frac{2\varepsilon}{3}})}.\end{equation}

and let
\begin{equation}
\varphi(x) := \frac{\text{dist} (x, E \setminus N_{2r}(\mathcal{M}(A_1, A_2, \rho, n)))}{\text{dist} (x, E \setminus N_{2r}(\mathcal{M}(A_1, A_2, \rho, n))) + \text{dist} (x, N_r(\mathcal{M}(A_1, A_2, \rho, n)))}.
\end{equation}
and consider the vector field

\[ W(x) := \begin{cases} 
-\varepsilon f(x) \varphi(x) \frac{\nabla I(x)}{\|I(x)\|} & \text{for } I'(x) \neq 0 \\
0 & \text{for } I'(x) = 0
\end{cases} \quad (3.238) \]

Notice that since \( n \in \mathcal{N}(2r, \rho) \), by (3.98), there is a \( \hat{\delta} > 0 \) such that \( \|I'(z)\| \geq \hat{\delta} \) for all \( z \in N_2r(\mathcal{M}(A_1, A_2, \rho, n)) \). Notice that \( \hat{\delta} \) could depend on \( n \) and \( \rho \), in contrast to \( \delta(A_1, A_2, r) \). However, since \( \rho < \rho^*(2r) \) and \( n \in \mathcal{N}(2r, \rho) \), by Proposition 3.1.20, we have \( \|I'(z)\| \geq \delta(A_1, A_2, r) \) for all \( z \in N_r(\mathcal{M}(A_1, A_2, \rho, n)) \backslash N_{r/8}(\mathcal{M}(A_1, A_2, \rho, n)) \).

However, it is more convenient to use \( \hat{\delta} \) to show that \( W(x) \) is bounded. In particular, for \( x \in N_2r(\mathcal{M}(A_1, A_2, \rho, n)) \), we know that

\[ \|W(x)\| \leq \frac{\varepsilon}{\hat{\delta}}. \quad (3.239) \]

Next, we claim that \( W(x) \) is locally Lipschitz. To show this, it suffices to show that if \( I'(x) = 0 \), there is a neighborhood of \( x \) on which \( f(x)\varphi(x) = 0 \). If this is not the case, there is an \( x \) with \( I'(x) = 0 \) and a sequence \( x_n \to x \) as \( n \to \infty \) with \( f(x_n)\varphi(x_n) > 0 \). But by (3.236), if \( \varphi(x_n) \neq 0 \), then we must have \( x_n \in N_2r(\mathcal{M}(A_1, A_2, \rho, n)) \), and thus \( x \in \overline{N_2r(\mathcal{M}(A_1, A_2, \rho, n))} \), hence \( I'(x) \neq 0 \).

Next, we claim that \( W(x) \) is bounded. If \( x \notin N_2r(\mathcal{M}(A_1, A_2, \rho, n)) \), then \( \varphi(x) = 0 \) and so \( W(x) = 0 \). Therefore, (3.239) gives us a bound on \( W(x) \). Notice that this implies that the solution \( \psi(s, x) \) of

\[ \frac{d\psi}{ds} = W(\psi), \quad \psi(0, x) = 0 \quad (3.240) \]

exists for all \( s \geq 0 \). Next, we claim that

if \( x \in \overline{N_{r/2}(\mathcal{M}(A_1, A_2, \rho, n))} \) and \( I(x) \leq \bar{c} + \frac{2\varepsilon}{3} \), then there exists an \( 0 \leq s(x) < \infty \) such that \( I(\psi(s(x), x)) \leq \bar{c} - \frac{\varepsilon}{3} \). \quad (3.241)
We have two cases to consider: either

(a) \( \psi(s, x) \in N_r(\mathcal{M}(A_1, A_2, \rho, n)) \) for all \( s \geq 0 \) or

(b) there is a maximal interval \((s_1, s_2)\) such that \( \psi([0, s_2], x) \subset N_r(\mathcal{M}(A_1, A_2, \rho, n)), \psi((s_1, s_2), x) \subset N_r(\mathcal{M}(A_1, A_2, \rho, n)) \setminus N_{r/2}(\mathcal{M}(A_1, A_2, \rho, n)) \), and \( \psi(s, x) \) travels from \( \partial N_{r/2}(\mathcal{M}(A_1, A_2, \rho, n)) \) to \( \partial N_r(\mathcal{M}(A_1, A_2, \rho, n)) \) as \( s \) goes from \( s_1 \) to \( s_2 \).

Suppose that (a) happens, and \( \psi(s, x) \) never reaches \( \partial I_{\bar{c}} - \varepsilon \). Then, by (3.236) and (3.237), \( f(\psi(s, x)) \varphi(\psi(s, x)) = 1 \) for all \( s \geq 0 \), and thus

\[
I(\psi(s, x)) = I(x) + \int_0^s \frac{d}{ds} I(\psi(s, x)) ds \\
= I(x) - \varepsilon \int_0^s I'(\psi(s, x)) \frac{\nabla I(\psi(s, x))}{\|I'(\psi(s, x))\|^2} f(\psi(s, x)) \varphi(\psi(s, x)) ds \\
= I(x) - \varepsilon s \rightarrow -\infty
\]

as \( s \rightarrow \infty \). Thus, if (a) occurs, then eventually \( \psi(s, x) \) reaches \( \partial I_{\bar{c}} - \frac{\varepsilon}{\lambda} \). Suppose now that (b) occurs, and \( \psi(s, x) \) reaches \( \partial N_r(\mathcal{M}(A_1, A_2, \rho, n)) \) before \( \partial I_{\bar{c}} - \frac{\varepsilon}{\lambda} \). By (3.236) and (3.237), we will then have \( f(\psi(s, x)) \varphi(\psi(s, x)) = 1 \) for all \( s \in [s_1, s_2] \).

We will also have

\[
\varepsilon > I(\psi(s_1, x)) - I(\psi(s_2, x)) = \int_{s_2}^{s_1} \frac{d}{ds} I(\psi(s, x)) ds \\
= \int_{s_1}^{s_2} \varepsilon ds \\
= \varepsilon(s_2 - s_1)
\]
Thus, \( s_2 - s_1 < 1 \). Next, by Proposition 3.1.20, we will have

\[
\frac{r}{2} \leq \left\| \psi(s_2, x) - \psi(s_1, x) \right\| \leq \varepsilon \int_{s_1}^{s_2} \frac{1}{\|I'(\psi(s, x))\|} ds \\
\leq \frac{\varepsilon (s_2 - s_1)}{\delta(A_1, A_2, r)}
\]

(3.244)
since \( n \in \mathcal{N}(2r, \rho) \) and for \( s \in (s_1, s_2) \),

\[
\psi(s, x) \in N_r(\mathcal{M}(A_1, A_2, \rho, n)) \setminus N_{\frac{1}{2}}(\mathcal{M}(A_1, A_2, \rho, n)) \\
\subset N_r(\mathcal{M}(A_1, A_2, \rho, n)) \setminus N_{\frac{1}{2}}(\mathcal{M}(A_1, A_2, \rho, n)).
\]

(3.245)

Combining (3.243) and (3.244), we see that

\[
\frac{r \delta(A_1, A_2, r)}{2} < \varepsilon,
\]

(3.246)
contradicting (3.200). Thus, (3.241) is proved, and \( \psi(s, x) \) must reach \( \partial I_{c^{-\frac{2}{3}}} \) before reaching \( \partial N_r(\mathcal{M}(A_1, A_2, \rho, n)) \) for all \( x \in N_{\frac{1}{2}}(\mathcal{M}(A_1, A_2, \rho, n)) \cap (I)^{\frac{\varepsilon + \frac{2\rho}{3}}{3}} \). Now, consider the map \( \sigma : N_{\frac{1}{2}}(\mathcal{M}(A_1, A_2, \rho, n)) \cap (I)^{\frac{\varepsilon + \frac{2\rho}{3}}{3}} \rightarrow [0, \infty) \) given by

\[
\sigma(x) := \inf \left\{ s \geq 0 \mid I(\psi(s, x)) \leq \bar{c} - \frac{\varepsilon}{3} \right\}
\]

(3.247)
Notice that by (3.241), \( \sigma(x) \) is always finite for \( x \in N_{\frac{1}{2}}(\mathcal{M}(A_1, A_2, \rho, n)) \cap (I)^{\frac{\varepsilon + \frac{2\rho}{3}}{3}} \). Since \( f(x)\varphi(x)I'(x) \neq 0 \) for all \( x \in N_r(\mathcal{M}(A_1, A_2, \rho, n)) \cap (I)^{\frac{\varepsilon + \frac{2\rho}{3}}{3}} \), if we follow the proof of the analogous portion of Proposition 3.1.10 (see (3.41)-(3.43)), we see that the function \( \sigma : N_{\frac{1}{2}}(\mathcal{M}(A_1, A_2, \rho, n)) \cap (I)^{\frac{\varepsilon + \frac{2\rho}{3}}{3}} \rightarrow [0, \infty) \) is continuous. Next, we show that

\[
\|\psi(\sigma(x), x) - x\| \leq 2r
\]

(3.248)
for all \( x \in N_{\frac{1}{2}}(\mathcal{M}(A_1, A_2, \rho, n)) \cap (I)^{\frac{\varepsilon + \frac{2\rho}{3}}{3}} \). We again consider the two cases, (a) and (b). If (a) occurs, then \( \psi(\sigma(x), x) \in N_r(\mathcal{M}(A_1, A_2, \rho, n)) \), and so (3.248) is
satisfied. Suppose then that (b) occurs. We showed that \( \psi(s, x) \) must reach \( \partial I^c - \varepsilon \) before reaching \( \partial N_r(M(A_1, A_2, \rho, n)) \). Thus, \( \sigma(x) \leq s_2 \), and therefore \( \psi(\sigma(x), x) \in \overline{N_r(M(A_1, A_2, \rho, n))} \). Since \( x \) is assumed to be in \( \overline{N_r^2(M(A_1, A_2, \rho, n))} \), we again have (3.248).

Now, if \( I(g_{k, \rho, n}(\theta_1, \theta_2)) > \bar{c} - \frac{\varepsilon}{3} \), then Lemma 3.2.4 implies that \( g_{k, \rho, n}(\theta_1, \theta_2) \in N_2^r(M(A_1, A_2, \rho, n)) \). Moreover, by (3.200), Lemma 3.2.2 and (3.216), we will have

\[
I(g_{k, \rho, n}(\theta_1, \theta_2)) = \sum_{i=1,2} \left( \tilde{J}_q(\hat{c}_{i,n}(\theta_i) + q_{i_A_i, \rho} - q_i) + I(q_i) \right) \\
\leq \sum_{i=1,2} \left( \tilde{J}_q(\gamma_i(\theta_i)) + I(q_i) \right) + \frac{\varepsilon}{6} \\
\leq \sum_{i=1,2} \left( \hat{c}_i + I(q_i) \right) + \frac{\varepsilon}{2} + \frac{\varepsilon}{6} = \bar{c} + \frac{2\varepsilon}{3}
\]

(3.249)

Thus, using (3.249), we may define

\[
G(\theta_1, \theta_2) := \begin{cases} \\
\psi(\sigma(g_{k, \rho, n}(\theta_1, \theta_2)), g_{k, \rho, n}(\theta_1, \theta_2)) & \text{if } g_{k, \rho, n}(\theta_1, \theta_2) \in N_2^r(M(A_1, A_2, \rho, n)) \cap (I)^{\bar{c} + \frac{2\varepsilon}{3}} \\
g_{k, \rho, n}(\theta_1, \theta_2) & \text{if } g_{k, \rho, n}(\theta_1, \theta_2) \notin N_2^r(M(A_1, A_2, \rho, n)) \cap (I)^{\bar{c} + \frac{2\varepsilon}{3}} \\
\end{cases}
\]

(3.250)

\[
G(\theta_1, \theta_2) := \begin{cases} \\
\psi(\sigma(g_{k, \rho, n}(\theta_1, \theta_2)), g_{k, \rho, n}(\theta_1, \theta_2)) & \text{if } g_{k, \rho, n}(\theta_1, \theta_2) \in N_2^r(M(A_1, A_2, \rho, n)) \\
g_{k, \rho, n}(\theta_1, \theta_2) & \text{if } g_{k, \rho, n}(\theta_1, \theta_2) \notin N_2^r(M(A_1, A_2, \rho, n)) \\
\end{cases}
\]

(3.250)

Notice that

\[
||G(\theta) - g_{k, \rho, n}(\theta)|| \leq 2r,
\]

(3.251)
for all $\theta \in [0, 1]^2$ since if $g_{\kappa, \rho, n}(\theta) \not\in N_2(\mathcal{M}(A_1, A_2, \rho, n))$, then $G(\theta) = g_{\kappa, \rho, n}(\theta)$, while for $g_{\kappa, \rho, n} \in N_2(\mathcal{M}(A_1, A_2, \rho, n))$, (3.248) implies (3.251). We claim now that $G(\theta_1, \theta_2) \in C([0, 1]^2, E)$. To see this, suppose $\theta_m \to \bar{\theta}$ in $[0, 1]^2$ as $m \to \infty$. Since $N_2(\mathcal{M}(A_1, A_2, \rho, n))$ is open, if $g_{\kappa, \rho, n}(\bar{\theta}) \in N_2(\mathcal{M}(A_1, A_2, \rho, n))$, then the continuity of $g_{\kappa, \rho, n}$ implies that $G(\theta_m) \to G(\bar{\theta})$ as $m \to \infty$. Similarly, if $g_{\kappa, \rho, n}(\bar{\theta})$ is in the interior of $E \setminus N_2(\mathcal{M}(A_1, A_2, \rho, n))$, then the continuity of $g_{\kappa, \rho, n}$ implies that $G(\theta_m) \to G(\bar{\theta})$ as $m \to \infty$. Suppose now that $g_{\kappa, \rho, n}(\bar{\theta}) \in \partial N_2(\mathcal{M}(A_1, A_2, \rho, n))$. If $g_{\kappa, \rho, n}(\theta_m) \in E \setminus N_2(\mathcal{M}(A_1, A_2, \rho, n))$ for all $m$, the continuity of $g_{\kappa, \rho, n}$ again guarantees that $G(\theta_m) = g_{\kappa, \rho, n}(\theta_m) \to g_{\kappa, \rho, n}(\bar{\theta}) = G(\bar{\theta})$ as $m \to \infty$. Suppose now that $g_{\kappa, \rho, n}(\theta_m) \in N_2(\mathcal{M}(A_1, A_2, \rho, n))$ for all $m$. Notice that since $g_{\kappa, \rho, n}(\bar{\theta}) \in \partial N_2(\mathcal{M}(A_1, A_2, \rho, n))$, Lemma 3.2.4 implies that $I(g_{\kappa, \rho, n}(\bar{\theta})) \leq \bar{c} - \frac{\varepsilon}{3}$, and so by (3.247) we must have $\sigma(g_{\kappa, \rho, n}(\bar{\theta})) = 0$. Then, since $\sigma : N_2(\mathcal{M}(A_1, A_2, \rho, n)) \cap I^{c - \frac{\varepsilon}{2}} \to [0, \infty)$ is continuous, we must have $\sigma(g_{\kappa, \rho, n}(\theta_m)) \to 0$ as $m \to \infty$. But then the continuity of $g_{\kappa, \rho, n}, \psi$ and $\sigma$ will imply that

$$G(\theta_m) = \psi(\sigma(g_{\kappa, \rho, n}(\theta_m)), g_{\kappa, \rho, n}(\theta_m))$$

$$\to \psi(0, g_{\kappa, \rho, n}(\bar{\theta})) = g_{\kappa, \rho, n}(\bar{\theta})$$

(3.252)

as $m \to \infty$. Next, notice that

$$\max_{\theta \in [0, 1]^2} I(G(\theta)) \leq \bar{c} - \frac{\varepsilon}{3}.$$  

(3.253)

To see this, if $g_{\kappa, \rho, n}(\theta) \not\in N_2(\mathcal{M}(A_1, A_2, \rho, n))$, then $I(g_{\kappa, \rho, n}(\theta)) \leq \bar{c} - \frac{\varepsilon}{3}$ by Lemma 3.2.4, and for such $\theta$, $G(\theta) = g_{\kappa, \rho, n}(\theta)$. Next, if $g_{\kappa, \rho, n}(\theta) \in N_2(\mathcal{M}(A_1, A_2, \rho, n))$,
then by (3.249), we have \( G(\theta) = \psi(\sigma(g_{\kappa,\rho,n}(\theta)), g_{\kappa,\rho,n}(\theta)) \). But (3.247) implies
\[
I(\psi(\sigma(g_{\kappa,\rho,n}(\theta)), g_{\kappa,\rho,n}(\theta))) \leq \bar{c} - \varepsilon^3, \text{ and so (3.253) is proved.}
\]
Because \( I(g_{\kappa,\rho,n}(0, \theta_2)) < \bar{c} - \varepsilon^3 \), we know that Lemma 3.2.4 implies that
\( g_{\kappa,\rho,n}(0, \theta_2) \notin N_{\frac{1}{2}}(M(A_1, A_2, \rho, \kappa)) \). Thus, by (3.250), we must have
\[
G(0, \theta_2) = g_{\kappa,\rho,n}(0, \theta_2). \tag{3.254}
\]
for all \( \theta_2 \in [0, 1] \). Similarly, \( G(1, \theta_2) = g_{\kappa,\rho,n}(1, \theta_2) \), \( G(\theta_1, 0) = g_{\kappa,\rho,n}(\theta_1, 0) \) and
\( G(\theta_1, 1) = g_{\kappa,\rho,n}(\theta_1, 1) \) for all \( \theta_1, \theta_2 \in [0, 1] \). Notice that this means that
\[
G(0, \theta_2)(t) = \xi \text{ for all } t \in [\omega(A_1, \rho) + 1, \alpha(A_2, \rho) + n - 1], \text{ and all } \theta_2 \in [0, 1]. \tag{3.255}
\]
and similarly for the other “boundaries”. If we knew that in fact \( G(\theta_1, \theta_2)(t) = \xi \) for all \( t \in [\omega(A_1, \rho) + 1, \alpha(A_2, \rho) + n - 1] \), then because of (3.255), we would have
\( G \in \Gamma(d_0) \). However, it seems very unlikely that this would happen, and so we must work harder. We do know that
\[
\|\xi - G(\theta_1, \theta_2)\|_{L^\infty(\omega(A_1, \rho) + 1, \alpha(A_2, \rho) + n - 1)} \leq C\|g_{\kappa,\rho,n}(\theta_1, \theta_2) - G(\theta_1, \theta_2)\| \leq \frac{\eta}{2} \tag{3.256}
\]
by our choice of \( r \). For notational convenience, we define
\[
S_{n,\rho} := [\omega(A_1, \rho) - 1, \alpha(A_2, \rho) + n - 1] \tag{3.257}
\]
With this in mind, we replace \( G(\theta_1, \theta_2) \) on the interval \( S_{n,\rho} \) by the minimizer of
\[
A(q) := \int_{S_{\kappa,\rho}} \frac{1}{2} |\dot{q}|^2 - V(t, q) dt \tag{3.258}
\]
over the set

\[ A(\theta_1, \theta_2) := \{ u \in W^{1,2}(S_{n,\rho}) \mid u(\partial S_{n,\rho}) = G(\theta_1, \theta_2)(\partial S_{n,\rho}) \text{ and } \|u - \xi\|_{L^\infty(S_n)} \leq \eta/2 \} \]

(3.259)

Notice that by our assumptions about the potential \( V \), \( A \) is strictly convex on \( A(\theta_1, \theta_2) \), and so there is a unique minimizer \( q_{\theta_1,\theta_2} \) for each \((\theta_1, \theta_2) \in [0, 1]^2\). We claim now that in fact the minimizer solves (HS). To see this, it suffices to show that

\[ \|q_{\theta_1,\theta_2} - \xi\|_{L^\infty(S_{n,\rho})} < \eta/2 \]

(3.260)

Notice that we must have

\[ \|q_{\theta_1,\theta_2} - \xi\|_{L^\infty(S_{n,\rho})} \leq C\|q_{\theta_1,\theta_2} - \xi\|_{W^{1,2}(S_{n,\rho})}, \]

(3.261)

By our assumptions about \( V \), there exist \( 0 < \beta_1 \leq \beta_2 \) such that if \( x \in B_{\eta/2}(\xi) \), then \( \beta_1|x - \xi|^2 \leq -V(t, x) \leq \beta_2|x - \xi|^2 \). Thus,

\[ \min \left\{ \frac{1}{2}, \beta_1 \right\} \|q_{\theta_1,\theta_2} - \xi\|_{W^{1,2}(S_{n,\rho})}^2 \leq A(q_{\theta_1,\theta_2}) \leq A(G(\theta_1, \theta_2)) \leq \max \left\{ \frac{1}{2}, \beta_2 \right\} \|G(\theta_1, \theta_2) - \xi\|_{W^{1,2}(S_{n,\rho})}^2 \leq 4 \max \left\{ \frac{1}{2}, \beta_2 \right\} r^2. \]

(3.262)

Combining (3.261) and (3.262), we have

\[ \|q_{\theta_1,\theta_2} - \xi\|_{L^\infty(S_{n,\rho})} \leq 2C \sqrt{\max \left\{ \frac{1}{2}, \beta_2 \right\} \min \left\{ \frac{1}{2}, \beta_1 \right\} r} < \eta/2 \]

(3.263)
by our choice of \( r \). Thus, \( q_{\theta_1, \theta_2} \) satisfies (HS). Next, we show that the minimizer depends continuously on \((\theta_1, \theta_2) \in [0, 1]^2\).

Suppose that \((\theta_{1,m}, \theta_{2,m}) \to (\theta_1, \theta_2) \) as \( m \to \infty \). Notice that by (3.262), \( A(q_{\theta_1, \theta_2, m}) \) is bounded independently of \((\theta_{1,m}, \theta_{2,m}) \in [0, 1]^2\). But then following (3.262)

\[
\min \left\{ \frac{1}{2}, \beta_1 \right\} \| q_{\theta_1, \theta_2, m} - \xi \|_{W^{1,2}(S_n, \rho)} \leq A(q_{\theta_1, \theta_2, m}),
\]

(3.264)

and so \( \| q_{\theta_1, \theta_2, m} \|_{W^{1,2}(S_n, \rho)} \) is bounded. But then on a subsequence \( q_{\theta_1, \theta_2, m_j} \) converges weakly to \( q \) in \( W^{1,2}(S_n, \rho) \). We claim that in fact \( q \) satisfies (HS). To see this, notice that \( A'(q_{\theta_1, \theta_2, m}) = 0 \) for all \( n \), since \( q_{\theta_1, \theta_2, m} \) satisfies (HS). Thus, for any \( \varphi \in C_c^\infty(S_n, \rho) \), we will have

\[
0 = A'(q_{\theta_1, \theta_2, m_j}) \varphi = \int_{S_n, \rho} \langle \dot{q}_{\theta_1, \theta_2, m_j}, \dot{\varphi} \rangle - \langle V_q(t, q_{\theta_1, \theta_2, m_j}), \varphi \rangle \ dt.
\]

(3.265)

Since \( q \) is the weak limit of \( q_{\theta_1, \theta_2, m_j} \), \( q_{\theta_1, \theta_2, m_j} \to q \) in \( L^\infty(S_n, \rho) \) and \( \dot{q}_{\theta_1, \theta_2, m_j} \) converges weakly to \( \dot{q} \) in \( L^2(S_n, \rho) \). Therefore,

\[
\int_{S_n, \rho} \langle \dot{q}, \dot{\varphi} \rangle \ dt = \lim_{j \to \infty} \int_{S_n, \rho} \langle \dot{q}_{\theta_1, \theta_2, m_j}, \dot{\varphi} \rangle \ dt 
\]

(3.266)

and

\[
\int_{S_n, \rho} \langle V_q(t, q), \varphi \rangle \ dt = \lim_{j \to \infty} \int_{S_n, \rho} \langle V_q(t, q_{\theta_1, \theta_2, m_j}), \varphi \rangle \ dt.
\]

(3.267)

Combining (3.265) - (3.267), we see that

\[
0 = \int_{S_n, \rho} \langle \dot{q}, \dot{\varphi} \rangle - \langle V_q(t, q), \varphi \rangle \ dt.
\]

(3.268)

Since this is true for all \( \varphi \in C_c^\infty(S_n, \rho) \), \( q \) solves (HS). Next, we show that in fact \( q_{\theta_1, \theta_2, m_j} \to q \) as \( m \to \infty \) in \( C^2(S_n, \rho) \). This will follow from (HS). In particular,
we must have

\[ \ddot{q}_{\theta_1, n_j, \theta_2, n_j} = V_q(t, q_{\theta_1, m_j, \theta_2, m_j}) \to V_q(t, q) = \ddot{q} \quad (3.269) \]

in \( L^\infty(S_{n, \rho}) \). Then, a simple interpolation inequality implies that

\[ \dot{q}_{\theta_1, m_j, \theta_2, m_j} \to \dot{q} \quad (3.270) \]

in \( L^\infty(S_{n, \rho}) \) as \( m \to \infty \), and so \( q_{\theta_1, m_j, \theta_2, m_j} \to q \) in \( C^2(S_{n, \rho}) \). Now, because \( q_{\theta_1, m_j, \theta_2, m_j}(\partial S_{n, \rho}) \to q_{\theta_1, \theta_2}(\partial S_{n, \rho}) \) as \( m \to \infty \), we must have \( q(\partial S_{n, \rho}) = q_{\theta_1, \theta_2}(\partial S_{n, \rho}) \). Moreover, since \( q_{\theta_1, m_j, \theta_2, m_j} \to q \) in \( C^2(S_{n, \rho}) \), we must have \( A(q_{\theta_1, m_j, \theta_2, m_j}) \to A(q) \) as \( m \to \infty \). In particular, by (3.263), we will have

\[ \|q - \xi\|_{L^\infty(S_{n, \rho})} \leq 2C \sqrt{\max\left\{ \frac{1}{2}, \beta_2 \right\} \min\left\{ \frac{1}{2}, \beta_1 \right\} r < \frac{\eta}{2} \quad (3.271) \]

by our choice of \( r \). Thus, \( q \in A(\theta_1, \theta_2) \). We claim now that \( q \) is a minimizer of \( A \) in \( A(\theta_1, \theta_2) \), and therefore \( q = q_{\theta_1, \theta_2} \). If \( q \) is not a minimizer, then (since \( A \) is strictly convex) there is a \( u \in C^\infty_c(S_{n, \rho}) \) such that \( u + q \in \text{int}A(\theta_2, \theta_2) \) (where int denotes the interior) and \( A(u + q) < A(q) \). Since each \( q_{\theta_1, m_j, \theta_2, m_j} \) is in the interior of \( A(\theta_1, m_j, \theta_2, m_j) \), we must have \( u + q_{\theta_1, m_j, \theta_2, m_j} \in A(\theta_1, m_j, \theta_2, m_j) \) for all large \( j \). Thus, we will have

\[ A(q_{\theta_1, m_j, \theta_2, m_j}) \leq A(q_{\theta_1, m_j, \theta_2, m_j} + u) \quad (3.272) \]

since \( q_{\theta_1, m_j, \theta_2, m_j} \) is a minimizer of \( A \) in \( A(\theta_1, m_j, \theta_2, m_j) \). Now, since \( q_{\theta_1, m_j, \theta_2, m_j} \to q \) in \( C^2(S_{n, \rho}) \) as \( j \to \infty \), passing to the limit in (3.272) as \( j \to \infty \) gives us

\[ A(q) \leq A(q + u) < A(q), \quad (3.273) \]
which is impossible. Thus, $q$ is a minimizer and so $q = q_{\theta_1, \theta_2}$. Now, suppose that there is a subsequence $m_k$ for which $q_{\theta_1, m_k, \theta_2, m_k} \not\rightarrow q$. Arguing as above, there must be a solution $\tilde{q}$ of (HS) such that $\tilde{q}$ is the limit of $\theta_1, m_k, \theta_2, m_k$ in $C^2(S, \rho)$, and $\tilde{q} = q_{\theta_1, \theta_2} = q$. Therefore, every subsequence of $q_{\theta_1, m, \theta_2, m}$ must converge to $q_{\theta_1, \theta_2}$, and the continuity of the minimizers depending on the boundary data is proved, and hence we have continuity in $\theta_1, \theta_2$. Thus, if we take

$$\bar{G}(\theta_1, \theta_2, t) := \begin{cases} G(\theta_1, \theta_2, t) & \text{for } t \leq \omega(A_1, \rho) + 1 \\ q_{\theta_1, \theta_2}(t) & \text{for } t \in (\omega(A_1, \rho) + 1, \alpha(A_2, \rho) + n - 1) = S, \\ G(\theta_1, \theta_2, t) & \text{for } t \geq \alpha(q_2, \rho) + n - 1 \end{cases}$$

(3.274)

then $G \in C([0, 1], E)$ and we must have

$$\max_{(\theta_1, \theta_2) \in [0, 1]^2} I(\tilde{G}(\theta_1, \theta_2)) \leq \max_{(\theta_1, \theta_2) \in [0, 1]^2} I(G(\theta_1, \theta_2)) \leq \bar{c} - \frac{\varepsilon}{3}$$

(3.275)

Because $G(0, \theta_2)(t) = g_{\kappa, \rho, n}(0, \theta_2)(t)$ by (3.254) and $g_{\kappa, \rho, n}(0, \theta_2)(t) = \xi$ for all $t \in S$, we must have $q_{0, \theta_2}(t) = G(0, \theta_2)(t) = \xi$ for all $t \in S$. An analogous statement holds for $q_{1, \theta_2}$, $q_{0, \theta_1}$ and $q_{\theta_1, 1}$. Thus, we will have by (3.211)

$$\bar{G}(0, \theta_2) = G(0, \theta_2) = g_{\kappa, \rho, n}(0, \theta_2) \text{ for all } \theta_2 \in [0, 1]$$

(3.276)

Similarly,

$$\bar{G}(1, \theta_2) = G(1, \theta_2) = g_{\kappa, \rho, n}(1, \theta_2) \text{ for all } \theta_2 \in [0, 1]$$

$$\bar{G}(\theta_1, 0) = G(\theta_1, 0) = g_{\kappa, \rho, n}(\theta_1, 0) \text{ for all } \theta_1 \in [0, 1]$$

(3.277)

$$\bar{g}(\theta_1, 1) = G(\theta_1, 1) = g_{\kappa, \rho, n}(\theta_1, 1) \text{ for all } \theta_1 \in [0, 1].$$
For notational convenience, let

\[ \bar{t}_n := \text{midpoint of } S_{n, \rho} \]  

(3.278)

Next, we claim that as \( n \to \infty \),

\[ \| \bar{G}'(\theta_1, \theta_2) - \xi \|_{C^1([\bar{t}_n-2, \bar{t}_n+2])} \to 0. \]  

(3.279)

uniformly in \((\theta_1, \theta_2) \in [0, 1]^2\). Since \( N(2r, \rho) \) is infinite, we may assume that \( \omega(A_1, \rho) + 1 < \bar{t}_n - 2 \) and \( \bar{t}_n + 2 < \alpha(A_2, \rho) + n - 1 \). Thus, if we define

\[
G(\theta_1, \theta_2)(t) := \begin{cases} 
\bar{G}'(\theta_1, \theta_2)(t) & \text{for } t \leq \bar{t}_n - 2 \\
(t - (\bar{t}_n - 2))\xi + ((\bar{t}_n - 1) - t)\bar{G}(\theta_1, \theta_2)(\bar{t}_n - 2) & \text{for } t \in (\bar{t}_n - 2, \bar{t}_n - 1) \\
\xi & \text{for } t \in [\bar{t}_n - 1, \bar{t}_n + 1] \\
((\bar{t}_n + 2) - t)\xi + (t - (\bar{t}_n + 1))\bar{G}(\theta_1, \theta_2)(\bar{t}_n + 2) & \text{for } t \in (\bar{t}_n + 1, \bar{t}_n + 2) \\
\bar{G}'(\theta_1, \theta_2)(t) & \text{for } t \geq \bar{t}_n + 2 
\end{cases}
\]  

(3.280)

then \( G \in C([0, 1]^2, E) \). Moreover, for \( t \in (\bar{t}_n - 2, \bar{t}_n - 1) \), we know that

\[
| \bar{G}'(\theta_1, \theta_2)(t) - G(\theta_1, \theta_2)(t) | = | \bar{G}'(\theta_1, \theta_2)(t) - (t - (\bar{t}_n - 2))\xi \\
- ((\bar{t}_n - 1) - t)\bar{G}(\theta_1, \theta_2)(\bar{t}_n - 2) | \\
\leq | \bar{G}'(\theta_1, \theta_2)(t) - \xi | + | (t - (\bar{t}_n - 1))(\xi - \bar{G}(\theta_1, \theta_2)(\bar{t}_n - 2)) | \\
\leq 2 || \bar{G}'(\theta_1, \theta_2) - \xi ||_{L^\infty(\bar{t}_n-2, \bar{t}_n+2)}
\]  

(3.281)
for all \((\theta_1, \theta_2) \in [0, 1]^2\) and
\[
\left| \dot{G}(\theta_1, \theta_2)(t) - \dot{G}(\theta_1, \theta_2) \right| = \left| \dot{G}(\theta_1, \theta_2)(t) - (\xi - \ddot{G}(\theta_1, \theta_2)(\bar{t}_n - 2)) \right|
\leq \left| \frac{d}{dt} \left( \ddot{G}(\theta_1, \theta_2)(t) - \xi \right) \right| + \| \dddot{G}(\theta_1, \theta_2) - \xi \|_{L^\infty(\bar{t}_n - 2, \bar{t}_n + 2)}
\leq \| \dddot{G}(\theta_1, \theta_2) - \xi \|_{C^1(\bar{t}_n - 2, \bar{t}_n + 2)}
\] (3.282)

for all \((\theta_1, \theta_2) \in [0, 1]^2\). Combining (3.281) and (3.282), we see that if (3.279) holds, then
\[
\| \dddot{G}(\theta_1, \theta_2) - \xi \|_{C^1(\bar{t}_n - 2, \bar{t}_n - 1)} \to 0
\] (3.283)
uniformly in \((\theta_1, \theta_2) \in [0, 1]^2\) as \(n \to \infty\). Similarly, if (3.279) holds, then
\[
\| \dddot{G}(\theta_1, \theta_2) - \xi \|_{C^1(\bar{t}_n - 1, \bar{t}_n + 1)} \to 0
\] (3.284)
uniformly in \((\theta_1, \theta_2) \in [0, 1]^2\) as \(n \to \infty\). Next, if \(t \in (\bar{t}_n - 1, \bar{t}_n + 1)\), we will have
\[
\dddot{G}(\theta_1, \theta_2)(t) = \dddot{G}(\theta_1, \theta_2)(t) - \xi,
\] (3.285)
for all \((\theta_1, \theta_2) \in [0, 1]^2\) and so if (3.279) holds, then
\[
\| \dddot{G}(\theta_1, \theta_2) - \dddot{G}(\theta_1, \theta_2) \|_{C^1(\bar{t}_n - 1, \bar{t}_n + 1)} \to 0
\] (3.286)
uniformly in \((\theta_1, \theta_2) \in [0, 1]^2\) as \(n \to \infty\). Combining (3.283)-(3.286), we see that if (3.279) holds, then
\[
\| \dddot{G}(\theta_1, \theta_2) - \dddot{G}(\theta_1, \theta_2) \|_{C^1(\bar{t}_n - 2, \bar{t}_n + 2)} \to 0
\] (3.287)
as \(n \to \infty\). Thus, if (3.279) holds, we will have for all large \(n\)
\[
\max_{(\theta_1, \theta_2) \in [0, 1]^2} I(\dddot{G}(\theta_1, \theta_2)) \leq \bar{c} - \frac{\varepsilon}{6}
\] (3.288)
since (3.287) implies that \( \| \bar{G}(\theta_1, \theta_2) - G(\theta_1, \theta_2) \| \to 0 \) as \( n \to \infty \), uniformly in \((\theta_1, \theta_2) \in [0, 1]^2\).

We now show that \( G \in \Gamma(d_0) \), which will contradict Lemma 3.1.27. Clearly, if we take \( a_1 := \bar{t}_n - 1 \) and \( a_2 := \bar{t}_n + 1 \), we will have (i) of Definition 3.1.22 satisfied. Notice that with these values of \( a_1, a_2 \), we will have (as in (3.158) and (3.159))

\[
\bar{G}_1(\theta_1, \theta_2)(t) := \begin{cases} 
G(\theta_1, \theta_2)(t) & \text{for } t < a_1 \\
\xi & \text{for } t \geq a_1
\end{cases}
\]  

(3.289)

and

\[
\bar{G}_2(\theta_1, \theta_2)(t) := \begin{cases} 
\xi & \text{for } t < a_2 \\
G(\theta_1, \theta_2)(t) & \text{for } t \geq a_2
\end{cases}
\]  

(3.290)

It remains then to show that (ii) of Definition 3.1.22 is satisfied. By (3.289), we will have

\[
\bar{G}_1(0, \theta_2)(t) := \begin{cases} 
G(0, \theta_2)(t) & \text{for } t < a_1 \\
\xi & \text{for } t \geq a_1
\end{cases}
\]  

(3.291)

and thus by (3.280), we will have

\[
\bar{G}_1(0, \theta_2)(t) := \begin{cases} 
\bar{G}(0, \theta_2)(t) & \text{for } t \leq \bar{t}_n - 2 \\
(\bar{t}_n - 2)(t - (\bar{t}_n - 2))\xi + ((\bar{t}_n - 1) - t)\bar{G}(0, \theta_2)(\bar{t}_n - 2) & \text{for } t \in (\bar{t}_n - 2, \bar{t}_n - 1) \\
\xi & \text{for } t \geq \bar{t}_n - 1
\end{cases}
\]  

(3.292)

But by (3.276), we know that (3.292) becomes

\[
\bar{G}_1(0, \theta_2)(t) := \begin{cases} 
g_{\kappa, \rho, n}(0, \theta_2)(t) & \text{for } t \leq \bar{t}_n - 2 \\
(\bar{t}_n - 2)(t - (\bar{t}_n - 2))\xi + ((\bar{t}_n - 1) - t)g_{\kappa, \rho, n}(0, \theta_2)(\bar{t}_n - 2) & \text{for } t \in (\bar{t}_n - 2, \bar{t}_n - 1) \\
\xi & \text{for } t \geq \bar{t}_n - 1
\end{cases}
\]  

(3.293)
Now, since $\omega(A_1, \rho) + 1 < \bar{t}_n - 2$, we know that for $\omega(A_1, \rho) + 1 \leq t < \alpha(A_2, \rho) + n - 1$,
\[ g_{n,\rho,n}(\theta_1, \theta_2)(t) = \xi, \]
and so (3.293) becomes
\[
G_1(0, \theta_2)(t) := \begin{cases} 
  g_{n,\rho,n}(0, \theta_2)(t) & \text{for } t \leq \omega(A_1, \rho) + 1 \\
  \xi & \text{for } t \geq \omega(A_1, \rho) + 1 
\end{cases}
\]
(3.294)
\[
= \hat{\gamma}_{1,\kappa}(0)(t) + q_1_{A_1,\rho}(t) \quad \text{for } t \leq \omega(A_1, \rho) + 1
\]
\[
= \xi \quad \text{for } t \geq \omega(A_1, \rho) + 1
\]
\[
(3.295)
\]
since $q_1_{A_1,\rho}(t) = \xi$ for $t \geq \omega(A_1, \rho) + 1$, $\hat{\gamma}_{1,\kappa}(\theta_1)(t) = 0$ for $t \geq M(\kappa) + 1$ and we have chosen $\rho$ sufficiently small that $\omega(A_1, \rho) + 1 > M(\kappa) + 1$. Now, by (3.294) and (3.214), we have
\[
\|G_1(0, \theta_2) - q_1\| < d_0
\]
by Lemma 3.1.25 since $\rho < \rho^*(2r)$. Similarly, we will have
\[
G_1(1, \theta_2) = \hat{\gamma}_{1,\kappa}(1) + q_1_{A_1,\rho}.
\]
(3.296)
Then, by (3.214) we will have
\[
\|G_1(1, \theta_2) - \tau_1 q_1\| < d_0
\]
(3.297)
Finally, arguing exactly as above, we will have
\[
G_2(\theta_1, 0) = \tau_n(\hat{\gamma}_{2,\kappa}(0) + q_2_{A_2,\rho}) \quad \text{and}
\]
\[
G_2(\theta_1, 1) = \tau_n(\hat{\gamma}_{2,\kappa}(1) + q_2_{A_2,\rho})
\]
(3.298)
and so by (3.214) we will have
\[
\|G_2(\theta_1, 0) - \tau_n q_2\| < d_0
\]
\[
\|G_2(\theta_1, 1) - \tau_n \tau_1 q_2\| < d_0
\]
(3.299)
Therefore, (ii) of Definition 3.1.22 holds, and $G \in \Gamma(d_0)$.

Thus, to finish the proof, it remains only to show (3.279). To do this, we use the maximum principle to show that $|q_{\theta_1, \theta_2}(t) - \xi|^2$ is bounded uniformly in $(\theta_1, \theta_2)$ by a function $f_n$ such that $\|f_n\|_{(\bar{t}_n - 2, \bar{t}_n + 2)} \to 0$ as $n \to \infty$. Thus, $q_{\theta_1, \theta_2} - \xi$ will be $L^\infty$ small on $(\bar{t}_n - 2, \bar{t}_n + 2)$. Let $L$ be

$$L := -\frac{d^2}{dt^2} + a$$

(3.300)

where $a > 0$ is free for the moment. It can be shown that the unique solution of

$$Lf(t) = 0 \text{ for } t \in (\alpha, \beta) \quad f(\alpha) = \eta = f(\beta)$$

(3.301)

is given by

$$f(t) := \frac{\eta}{\sinh(\sqrt{a}(\beta - \alpha))} \left( \sinh(\sqrt{a}(\beta - t)) + \sinh(\sqrt{a}(t - \alpha)) \right)$$

(3.302)

Let $\bar{t} := \frac{1}{2}(\beta + \alpha)$ be the midpoint of $(\alpha, \beta)$. Now, we show that as $\beta - \alpha \to \infty$,

$$\|f\|_{L^\infty(\bar{t} - 2, \bar{t} + 2)} \to 0.$$  

(3.303)

By (3.302), it suffices to show that

$$\frac{\sinh(\sqrt{a}(\beta - t))}{\sinh(\sqrt{a}(\beta - \alpha))} \to 0 \text{ and }$$

(3.304)

$$\frac{\sinh(\sqrt{a}(t - \alpha))}{\sinh(\sqrt{a}(\beta - \alpha))} \to 0$$

(3.305)

uniformly for $t \in [\bar{t} - 2, \bar{t} + 2]$. We have

$$\frac{\sinh(\sqrt{a}(t - \alpha))}{\sinh(\sqrt{a}(\beta - \alpha))} = e^{-\sqrt{a}(\beta - t)} \frac{1 - e^{-2\sqrt{a}(t - \alpha)}}{1 - e^{-2\sqrt{a}(\beta - \alpha)}}.$$  

(3.306)

Now, since $\beta - \alpha \to \infty$, we will have $\alpha < \bar{t} - 2 < \bar{t} + 2 < \beta$ for all $\beta, \alpha$ with $\beta - \alpha$ sufficiently large. But then if $t \in [\bar{t} - 2, \bar{t} + 2]$, we will have $0 \leq t - \alpha$ and so
\[ e^{-2\sqrt{a}(t-\alpha)} \leq 1. \] Similarly, for all large \( \beta - \alpha \) large, we will have \( 1 - e^{-2\sqrt{a}(\beta-\alpha)} \geq \frac{1}{2} \).

Thus,

\[
\frac{(1 - e^{-2\sqrt{a}(t-\alpha)})}{(1 - e^{-2\sqrt{a}(\beta-\alpha)})}
\]

is bounded for all large \( \beta - \alpha \).

Next, notice that if \( t \in [\bar{t} - 2, \bar{t} + 2] \), then

\[
e^{-\sqrt{a}(\beta-t)} = e^{-\sqrt{a}t} e^{\sqrt{a}}
\]

\[
\leq e^{-\beta \sqrt{a} e^{\sqrt{a}}}
\]

\[
\leq e^{-\beta \sqrt{a} e^{(\bar{t}+2)\sqrt{a}}}
\]

\[
= e^{-\sqrt{a}\left(\frac{\beta-a}{2} + 2\right)} \to 0
\]

as \( \beta - \alpha \to \infty \). Combining (3.307) and (3.308), we see that (3.305) holds. A similar argument shows that (3.304) holds.

Now, we use the maximum principle to bound \( |\bar{G}(\theta)(t) - \xi|^2 \) by the solution of

\[
L f_n(t) = 0 \text{ for } t \in S_{n,\rho} \quad f_n(\partial S_{n,\rho}) = \eta
\]

(3.309)

Notice that \( \bar{G}(\theta)(t) = q_\theta(t) \) for \( t \in S_{n,\rho} \), and \( q_\theta \) is a solution of (HS). Thus,

\[
\frac{d}{dt} |\bar{G}(\theta)(t) - \xi|^2 = \langle \dot{\bar{G}}(\theta)(t), \bar{G}(\theta)(t) - \xi \rangle
\]

and so

\[
\frac{d^2}{dt^2} |\bar{G}(\theta)(t) - \xi|^2 = \langle \ddot{\bar{G}}(\theta)(t), \bar{G}(\theta)(t) - \xi \rangle + \left| \dot{\bar{G}}(\theta)(t) \right|^2
\]

\[
= \langle -V_q(t, \bar{G}(\theta)(t)), \bar{G}(\theta)(t) - \xi \rangle + \left| \dot{\bar{G}}(\theta)(t) \right|^2.
\]

(3.311)
But, by our assumptions about $V$, we know that there is a $\lambda > 0$ such that if $|x - \xi| < \eta/2$, then $\langle V_q(t, x), x - \xi \rangle \leq -\lambda |x - \xi|^2$. Thus, (3.311) implies that

$$L \left( |\bar{G}(\theta)(t) - \xi|^2 \right) = \langle V_q(t, \bar{G}(\theta)(t)), \bar{G}(\theta)(t) - \xi \rangle - \left| \dot{G}(\theta)(t) \right|^2 + a \left| \dot{G}(\theta) - \xi \right|^2 \leq (a - \lambda) \left| \bar{G}(\theta)(t) - \xi \right|^2 - \left| \dot{G}(\theta)(t) \right|^2 < 0 \tag{3.312}$$

if we take $a < \lambda$. Therefore,

$$L \left( f(t) - |\bar{G}(\theta)(t) - \xi|^2 \right) = Lf(t) - L \left( |\bar{G}(\theta)(t) - \xi|^2 \right) > 0 \tag{3.313}$$

for all $t \in S_{n,\rho}$. Since $|\bar{G}(\theta)(\partial S_{n,\rho}) - \xi| < \eta/2$, we will also have

$$f(\partial S_{n,\rho}) - |\bar{G}(\theta)(\partial S_{n,\rho}) - \xi|^2 \geq 0. \tag{3.314}$$

Thus, (3.313) and (3.314) and the Maximum Principle imply that

$$|\bar{G}(\theta)(t) - \xi|^2 \leq f(t) \tag{3.315}$$

for all $t \in S_{n,\rho}$. Notice that (3.315) is independent of $\theta$. Taking $\alpha := \omega(A_1, \rho)$ and $\beta := \alpha(A_2, \rho) + n - 1$, and noting that $n$ can be made arbitrarily large since we are assuming that $\mathcal{N}(2r, \rho)$ is infinite, (3.303) implies

$$\|\bar{G}(\theta) - \xi\|_{L^\infty(\tilde{t}_n-2,\tilde{t}_n+2)} \to 0 \tag{3.316}$$

as $n \to \infty$ uniformly in $\theta \in [0, 1]^2$. But then, since $\bar{G}(\theta)$ solves (HS), we must have

$$|\ddot{G}(\theta)(t)| = |V_q(t, \bar{G}(\theta)(t))|. \tag{3.317}$$

Now, by (3.316), since $V_q(t, \xi) \equiv 0$, (3.317) implies that

$$\|\ddot{G}(\theta)(t)\|_{L^\infty(\tilde{t}_n-2,\tilde{t}_n+2)} \to 0 \tag{3.318}$$
as $n \to \infty$, uniformly in $\theta \in [0, 1]^2$. Now, there is a constant $C$ such that

$$\|\dot{g}\|_{L^\infty(0, 4)} \leq C \left( \|\ddot{g}\|_{L^\infty(0, 4)} + \|g\|_{L^\infty(0, 4)} \right).$$

(3.319)

Therefore, (3.317)-(3.319) imply (3.279), which finishes the proof.

Notice that these methods may be extended to find solutions with any finite number of transitions between the equilibria, and such that on each transition, the solution is $W^{1,2}$ close to a solution of mountain pass type in $A_1$ (if the transition is from 0 to $\xi$) or in $A_2$ (if the transition is from $\xi$ to 0).
Chapter 4

Wells at Different Levels

4.1 Set-up

In this section, we turn our attention to the existence of homoclinic solutions to

\[(HS) \quad \ddot{q}(t) = -V_q(t, q(t))\]

in the context that the potential \( V \) has two wells of different heights. In particular, we assume that \( V \) satisfies:

(DV1) \( V \in C^3(\mathbb{R} \times \mathbb{R}, \mathbb{R}) \)

(DV2) \( V \) is 1 periodic in \( t \): \( V(t + 1, q) = V(t, q) \)

(DV3) \( q = 0 \) and \( q = 1 \) are non-degenerate local maxima of \( V \) for every \( t \). Moreover, we assume \( V(t, 0) = 0 \) and there exists a constant \( c_1 > 0 \) such that \( V(t, 1) \geq c_1 \) for all \( t \).

(DV4) \( q = \frac{1}{2} \) is a non-degenerate local minimum of \( V \), and there is a constant \( c_2 < 0 \) such that for every \( t \), \( V(t, \frac{1}{2}) \leq c_2 \).

(DV5) \( V_q(t, x) \neq 0 \) for \( x \in (1/2, 1) \), and there is a constant \( \Lambda > 0 \) such that \( V(t, x) \leq -\Lambda x^2 \) for \( 0 \leq x \leq 1/2 \).
There are constants $\alpha_0, \beta > 0$ such that $V(t, q) < -\alpha_0 q^2 + \beta$.

$V_{qq}, V_{tq}$ and $V_{qqq}$ are bounded.

For example, we could take $\tilde{V}(t, q) = -a(t)(q^4 - 2q^3 + q^2)$ for any smooth positive 1-periodic function $a$, and then modify for $|q|$ large to make sure that (DV7) is satisfied.

Notice that (DV3-4) imply that $q(t) \equiv 0, q(t) \equiv 1$ and $q(t) \equiv \frac{1}{2}$ are solutions to (HS). Instead of (DV5), we could assume that the only zeros of $V_q(t, x)$ occur for $x \in \{0, 1/2, 1\}$. Because of the deformation argument that we will use, we will only be concerned with $q(t)$ satisfying $0 \leq q(t) \leq 1$, which make (DV6) and (DV7) not as much of an imposition as they appear. The extra regularity that we require of $V$ is a technical assumption.

Our goal is to prove the following:

**Theorem 4.1.1.** If $V$ satisfies (DV1-7), then there is a solution $q$ of (HS) homoclinic to 0 such that there are exactly two points $a, b$ where $q(a) = \frac{1}{2} = q(b)$.

We will find such a solution by finding an appropriate sequence of subharmonic solutions, and then passing to the limit. Towards that end, we introduce a family of Hilbert spaces.

**Definition 4.1.2.**

$$E_k := W^{1,2}_{2k-per} = \{ q \in W^{1,2}_{loc}(\mathbb{R}) \mid q(t + 2k) = q(t) \}$$

equipped with the norm

$$\|q\|_{E_k}^2 := \int_{-k}^{k} (q^2 + q'^2) \, dt.$$
Let

$$I_k(q) := \int_{-k}^{k} \left( \frac{1}{2} \dot{q}^2 - V(t, q) \right) dt \text{ for } q \in E_k.$$ 

We collect some basic information about the functional $I_k$:

**Lemma 4.1.3.** $I_k$ is bounded from below by a constant depending only on $k$.

**Proof.** From (DV6), we have

$$I_k(q) = \int_{-k}^{k} \left( \frac{1}{2} \dot{q}^2 - V(t, q) \right) dt \geq \int_{-k}^{k} \left( \frac{1}{2} \dot{q}^2 + \alpha_0 q^2 - \beta \right) dt$$

$$= -2\beta k + \int_{-k}^{k} \left( \frac{1}{2} \dot{q}^2 + \alpha_1 q^2 \right) dt \quad (4.1)$$

$$\geq \min\{\frac{1}{2}, \alpha_1\} \|q\|^2_{E_k} - 2\beta k \geq -2\beta k$$

As a straightforward consequence, we show

**Corollary 4.1.4.** $I_k$ satisfies the (PS) condition.

**Proof.** If $I_k(a_n) \to c$, then $I_k(a_n)$ is bounded. But then $a_n$ is bounded in $E_k$ by (4.1), and so by standard theorems (see [17]), $I_k$ satisfies (PS). 

Under our assumptions, it can be shown that each $I_k$ satisfies the hypothesis of the Mountain Pass Theorem. Therefore,

$$c_k := \inf_{h \in \Gamma_k} \max_{s \in [0,1]} I_k(h_s) > 0$$

where

$$\Gamma_k := \{ h \in C([0,1], E_k) \mid h_0(t) = 0, \ h_1(t) = 1 \}$$
is a critical value for each $I_k$. With this in mind, we obtain a sequence $v_k$ of critical points of $I_k$. Each of these critical points satisfies (HS). It can be shown that this sequence is bounded in $C^2(\mathbb{R})$, and so there is a subsequence $v_{k_j}$ that converges to a solution to (HS) on $\mathbb{R}$. However, it is not clear how to show that the limit has the appropriate asymptotic behavior.

Thus, we are led to use a different sort of argument. In the proof of the Mountain Pass Theorem, a key ingredient is the existence of a deformation that decreases the functional at least a certain amount away from critical points. We will use another type of deformation, but not one arising from the gradient of the functional $I_k$. Instead, we will use a heat-like flow. This flow $\varphi_s(u)$ has the property that the number of intersections of $\varphi_s(u)$ with a solution of (HS) does not increase as $s$ increases. With this information, we can construct a sequence of subharmonic solutions that spend only a bounded amount of time between $\frac{1}{2}$ and 1. With this extra information, we are able to control the asymptotics of the limit.

Notice that on each $I_k$, we have a natural $\mathbb{Z}$ action. If $\tau_n q(t) := q(t - n)$, then $I_k(\tau_n q) = I_k(q)$ for any $q \in E_k, n \in \mathbb{Z}$.

### 4.2 An appropriate sequence of subharmonics

In this section, we construct the subharmonics whose limit will be the homoclinic. Before we do that, we need some technical lemmas. The first is an interpolation lemma that relates bounds on $\|\dot{u}\|_{L^\infty(\mathbb{R})}$ to bounds on $\|\ddot{u}\|_{L^\infty(\mathbb{R})}$ and $\|\dot{u}\|_{L^\infty(\mathbb{R})}$. 
Lemma 4.2.1. Suppose that $u \in C^2(\mathbb{R})$ and $\|\ddot{u}\|_{L^\infty(\mathbb{R})}, \|u\|_{L^\infty(\mathbb{R})} < \infty$. Then

$$\|\dot{u}\|_{L^\infty(\mathbb{R})} \leq 2\|u\|_{L^\infty(\mathbb{R})} + \|\ddot{u}\|_{L^\infty(\mathbb{R})}.$$ 

Proof. Fix an $x \in \mathbb{R}$. By the Mean Value Theorem, there is a $z$ between $x$ and $x + 1$ such that

$$\frac{u(x) - u(x + 1)}{x - (x + 1)} = \dot{u}(z). \quad (4.2)$$

But then

$$\dot{u}(x) - \dot{u}(z) = \int_x^z \ddot{u}(s) \, ds, \quad (4.3)$$

and hence

$$\dot{u}(x) = \dot{u}(z) + \int_x^z \ddot{u}(s) \, ds \quad (4.4)$$

$$= u(x) - u(x + 1) + \int_x^x \ddot{u}(s) \, ds. \quad (4.5)$$

Therefore

$$|\dot{u}(x)| \leq 2\|u\|_{L^\infty(\mathbb{R})} + \|\ddot{u}\|_{L^\infty(\mathbb{R})}|x - z| \quad (4.6)$$

$$\leq 2\|u\|_{L^\infty(\mathbb{R})} + \|\ddot{u}\|_{L^\infty(\mathbb{R})}. \quad (4.7)$$

Since this is true for any $x$, the result follows. \qed

We can also show the following:

Lemma 4.2.2. Suppose that $b - a \geq 2$ and $u \in C^2([a, b])$. Then

$$\|\dot{u}\|_{L^\infty([a, b])} \leq 2\|u\|_{L^\infty([a, b])} + \|\ddot{u}\|_{L^\infty([a, b])}.$$
Proof. The only difference with the proof above is that instead of using $x + 1$, we pick a $y \in [a, b]$ such that $|x - y| = 1$. Since $b - a \geq 2$, there is always such a $y$ for any $x \in [a, b]$. \qed

Now, we prove an estimate that will enable us to keep our subharmonics away from 0.

**Lemma 4.2.3.** Let

$$\Upsilon_k := \{I_k(q) : q \in E_k, \ 0 \leq q(t) \leq \frac{1}{2} \text{ for all } t, \text{ and } q(t) = \frac{1}{2} \text{ for some } t\},$$

and define

$$\alpha_k := \inf_{q \in \Upsilon_k} I_k(q).$$

There is a constant $a > 0$ such that $\alpha_k \geq a$ for all $k$.

Proof. Fix $k$, and consider a minimizing sequence $q_n \in \Upsilon_k$ such that $I_k(q_n) \to \alpha_k$. Because $0 \leq q_n \leq 1/2$, we know that $-V(t, q_n(t)) \geq 0$, hence

$$I_k(q_n) = \int_{-k}^{k} \frac{1}{2} \dot{q}_n^2 - V(t, q_n) \, dt \geq \frac{1}{2} \int_{-k}^{k} \dot{q}_n^2 \, dt \quad (4.8)$$

But then $\|\dot{q}_n\|_{L^2([-k, k])}$ is bounded. Since $q_n$ is also bounded in $L^\infty([-k, k])$ and thus in $L^2([-k, k])$, we know that (along a subsequence) $q_n$ converges weakly to some $q \in E_k$ and $\|q_n - q\|_{L^\infty([-k, k])} \to 0$. Therefore, $0 \leq q(t) \leq 1/2$. Next, for each $n$, there is a $t_n \in [-k, k]$ such that $q_n(t_n) = 1/2$. Since $[-k, k]$ is compact, it can be assumed that $t_n \to \hat{t}$. Now, we have

$$|q(\hat{t}) - 1/2| \leq |q(\hat{t}) - q_n(\hat{t})| + |q_n(\hat{t}) - q_n(t_n)| + |q_n(t_n) - 1/2|. \quad (4.9)$$

The last term is 0 and the first term goes to 0, because $q_n$ converges uniformly to $q$. The middle term also goes to 0, because $\|\dot{q}_n\|_{L^2([-k, k])}$ is bounded, and so $\{q_n\}$
is equicontinuous. Thus, \( q(\hat{t}) = 1/2 \), and thus \( I_k(q) \geq \alpha_k \). Finally, we can show that \( I_k \) is weakly lower semi-continuous, and thus \( I_k(q) \leq \alpha_k \). Therefore, we have a minimizer \( q \) of \( I_k \) in \( \Upsilon_k \) for every \( k \). Relabeling, we have a sequence \( a_k \in \Upsilon_k \) such that \( I_k(a_k) = \alpha_k \).

Finally, we can show that \( I_k \) is weakly lower semi-continuous, and thus \( I_k(q) \leq \alpha_k \). Therefore, we have a minimizer \( q \) of \( I_k \) in \( \Upsilon_k \) for every \( k \). Relabeling, we have a sequence \( a_k \in \Upsilon_k \) such that \( I_k(a_k) = \alpha_k \).

Suppose now that \( \alpha_k \) is not bounded away from 0. Then, we can find a subsequence such that \( \alpha_k \to 0 \). Thus, there is a sequence of \( a_k \) with \( 0 \leq a_k \leq 1/2 \), \( a_k(t_k) = 1/2 \) for some \( t_k \in [-k, k] \) and \( I_k(a_k) \to 0 \). By (DV5), there is a \( \Lambda \) such that \( -V(t, a_k(t)) \geq \Lambda a_k(t)^2 \) for all \( t \in [-k, k] \). But then

\[
\alpha_k = I_k(a_k) \geq \int_{-k}^{k} \left( \frac{1}{2} a_k^2 + \Lambda a_k^2 \right) dt \geq \min\{ \frac{1}{2}, \Lambda \} \|a_k\|^2_{E_k} \to 0. \tag{4.10}
\]

But this is impossible, since \( \|a_k\|_{L^\infty([-k,k])} = 1/2 \), \( \|\dot{a}_k\|^2_{L^2([-k,k])} \) is bounded by \( 2\alpha_k \) and \( \alpha_k \) is bounded (assuming \( \alpha_k \to 0 \)). \( \square \)

Next, we define a homotopy from 0 to 1 in each \( E_k \). We will use these homotopies to find a sequence \( q_k \) of subharmonic solutions of (HS) that are \( \geq 1/2 \) for an amount of time which is bounded independently of \( k \).

By (DV3), there is an \( \eta < 1 \) such that \( -V(t, \eta) \leq -\frac{\eta}{2} < 0 \) for all \( t \). Let \( \vartheta \in C_c^\infty(\mathbb{R}) \) be such that \( \vartheta(0) = \eta \geq \vartheta(t) \) and \( D^\alpha \vartheta(0) = 0 \) for all \( \alpha \geq 1 \). Moreover, we assume that \( \text{supp} \, \vartheta = [-\frac{1}{2}, \frac{1}{2}] \), \( \vartheta(t) = \vartheta(-t) \), and \( \dot{\vartheta}(t) < 0 \) for \( t \in (0, 1/2) \). The condition on the derivatives at 0 is not especially important, but it makes the pictures much nicer.

We use \( \vartheta \) to define a homotopy \( h_s^k \in C([0, 1], E_k) \) connecting \( u(t) \equiv 0 \) to \( u(t) \equiv 1 \) in \( E_k \) as \( s \) goes from 0 to 1.

For \( s \in [0, \frac{1}{2}] \), we let

\[
h_s^k(t) = 4s \vartheta(t).
\]
In particular, $h^k_s(t) = \vartheta(t)$. We have the following picture:

$$h^k_s(t) := \begin{cases} 
0 & \text{for } -k \leq t \leq 2(s - \frac{1}{4})(-k + \frac{1}{2}) - \frac{1}{2} \\
\vartheta(t - 2(s - \frac{1}{4})(-k + \frac{1}{2})) & \text{for } 2(s - \frac{1}{4})(-k + \frac{1}{2}) - \frac{1}{2} \leq t \leq 2(s - \frac{1}{4})(-k + \frac{1}{2}) \\
\vartheta(0) = \eta & \text{for } 2(s - \frac{1}{4})(-k + \frac{1}{2}) \leq t \leq 2(s - \frac{1}{4})(k - \frac{1}{2}) \\
\vartheta(t - 2(s - \frac{1}{4})(k - \frac{1}{2})) & \text{for } 2(s - \frac{1}{4})(k - \frac{1}{2}) \leq t \leq 2(s - \frac{1}{4})(k - \frac{1}{2}) + \frac{1}{2} \\
0 & \text{for } 2(s - \frac{1}{4})(k - \frac{1}{2}) + \frac{1}{2} \leq t \leq k 
\end{cases}$$

so $h^k_s$ looks like

Figure 1: $s \in (0, 1/4)$
Next, for $\frac{3}{4} < s \leq \frac{7}{8}$, we set

$$h^k_s(t) := \begin{cases} 
8(s - \frac{3}{4})\eta + (1 - 8(s - \frac{3}{4}))\vartheta(t + k - \frac{1}{2}) & \text{for } -k \leq t \leq -k + \frac{1}{2} \\
\vartheta(0) = \eta & \text{for } -k + \frac{1}{2} \leq t \leq k - \frac{1}{2} \\
8(s - \frac{3}{4})\eta + (1 - 8(s - \frac{3}{4}))\vartheta(t - k + \frac{1}{2}) & \text{for } k - \frac{1}{2} \leq t \leq k
\end{cases}$$

Figure 2: $s \in (1/4, 3/4)$
In particular, we have $h^k_{\frac{3}{8}}(t) = \eta$, and $h^k_s$ looks like:

![Graph](image1.png)

**Figure 3:** $s \in (3/4, 7/8)$

Finally, for $\frac{7}{8} < s \leq 1$, we put

$$h^k_s(t) = \eta + 8 \left( s - \frac{7}{8} \right)(1 - \eta).$$

and so $h^k_s$ looks like:

![Graph](image2.png)

**Figure 4:** $s \in (7/8, 1)$
Let
\[ J_s := \{ t \in [-k, k] \mid h^k_s(t) \neq 0 \}. \]

By construction, \( J_s \) is an interval centered around 0. Next, notice that for every \( s \), each \( h^k_s \) crosses 1/2 at most twice.

**Lemma 4.2.4.** There is a \( C \) (independent of \( k \)) such that \( I_k(h^k_s) \leq C \) for all \( s \in [0, 1] \).

**Proof.** Notice that because of the choice of \( \eta \), the maximum of \( I_k(h^k_s) \) does not occur for \( s \in \left[ \frac{7}{8}, 1 \right] \) (on this interval, \( I_k(h^k_s) \leq 0 \)). Let

\[
J^0_s := \text{intervals in } [-k, k] \text{ where } h^k_s(t) = 0 \quad (4.11)
\]

\[
J^\eta_s := \text{interval where } h^k_s(t) = \eta \quad (4.12)
\]

\[
J^{up}_s := \text{interval (of length } \frac{1}{2} \text{) where } \dot{h}^k_s(t) > 0 \quad (4.13)
\]

\[
J^{down}_s := \text{interval (of length } \frac{1}{2} \text{) where } \dot{h}^k_s(t) < 0 \quad (4.14)
\]

Notice that the length of the intervals in \( J^0_s \) plus the length of the interval in \( J^\eta_s \) is \( 2k - 1 \). Moreover, we have

\[
\int_{J^\eta_s} \left( \frac{1}{2}(\dot{h}^k_s)^2 - V(t, h^k_s) \right) dt = 0 \quad (4.15)
\]

and

\[
\int_{J^{up}_s} \left( \frac{1}{2}(\dot{h}^k_s)^2 - V(t, h^k_s) \right) dt = \int_{J^{down}_s} -V(t, \eta)dt \leq (-\frac{c_1}{2}) |J^\eta_s| < 0, \quad (4.16)
\]

where \( |J| \) denotes the length of the interval \( J \). Thus, it remains only to bound

\[
\int_{J^{up}_s} \left( \frac{1}{2}(\dot{h}^k_s)^2 - V(t, h^k_s) \right) dt
\]
\[ \int_{J_{\text{down}}} \left( \frac{1}{2} (\dot{h}_s^k)^2 - V(t, h_s^k) \right) dt. \]

For any \( s \), \( J_{\text{up}}^s \) is an interval of the form \((x, x + \frac{1}{2})\), hence

\[ \int_x^{x + \frac{1}{2}} \left( \frac{1}{2} (\dot{h}_s^k)^2 - V(t, h_s^k) \right) dt \leq \frac{1}{4} \| \dot{\vartheta} \|_{L^\infty}^2 + \frac{1}{2} \left( \max_{t \in [0,1], q \in [0,1]} |V(t, q)| \right) \]  

(4.17)

Notice that this last bound is independent of \( k \) and \( s \). A similar bound applies to

\[ \int_{J_{\text{down}}} \left( \frac{1}{2} (\dot{h}_s^k)^2 - V(t, h_s^k) \right) dt. \]

The flow we will use to find subharmonic solutions arises from the parabolic semilinear partial differential equation

\[(PDE) \quad w_s(s, t) = w_{tt}(s, t) + V_q(t, w(s, t)) \quad w(0, t) = u(t)\]

for some \( u \in E_k \). We have the following theorem:

**Theorem 4.2.5.**  
(1) For any \( u \in E_k \), there is a unique solution \( w(s, t) \) of \((PDE)\) such that \( s \mapsto w(s, \cdot) \) is in \( C([0, \infty), E_k) \) and \( w \) is a classical solution, in the sense that for every \( t \in \mathbb{R} \), \( s \mapsto w(s, t) \) is differentiable for every \( s > 0 \), and for every \( s > 0 \), \( t \mapsto w(s, t) \) is twice differentiable, and \((PDE)\) is satisfied pointwise.

(2) We define a flow on \( E_k \) by \( \varphi_s^k(u) := w(s, t) \), where \( w \) satisfies \((PDE)\), with initial condition \( u \). This satisfies the semi-group property: \( \varphi_{s_1 + s_2}^k(u) = \varphi_{s_1}^k(\varphi_{s_2}^k(u)) \) for any \( s_1, s_2 > 0 \).
(3) $\varphi_s^k(u)$ is continuous in $u$: for any $\varepsilon > 0$ and $T > 0$, there is a $\delta = \delta(\varepsilon, T, u) > 0$ such that if $\|u - v\|_{E_k} < \delta$, then $\|\varphi_s^k(u) - \varphi_s^k(v)\|_{E_k} < \varepsilon$ for all $0 \leq s \leq T$.

(4) For any sequence $s_i \to \infty$ and any $u \in E_k$, there is a subsequence $s_{i_j}$ and a solution $\bar{u} \in E_k$ of (HS) such that $\|\varphi_{s_{i_j}}^k(u) - \bar{u}\|_{E_k} \to 0$.

We postpone the proof of this theorem until the appendix. In order to simplify our notation, we suppress the $k$ and write $\varphi_s(\cdot)$. We will use the following lemma often. In effect, it says that $I_k$ is a Lyapunov function for the flow $\varphi_s$.

**Lemma 4.2.6.** Fix $k$ and $u \in E_k$. Then, the function $s \mapsto I_k(\varphi_s(u))$ is decreasing.

**Proof.** We postpone the proof until the appendix.

Next, a precise meaning can be given to the statement: the number of intersections of $\varphi_s(u)$ with a solution of (HS) is non-increasing as $s$ increases. Angenent has shown the following in [1]:

**Proposition 4.2.7.** Suppose that $0 \not\equiv u \in C^2((0, \infty) \times [-k, k])$ satisfies $u_s = u_{tt} + b(s,t)u$, $u(s,-k) = u(s,k)$ (periodic boundary conditions) and $|u(s,t)| \leq A$ where $b \in L^\infty([0, \infty) \times [-k, k])$. Then

(1) For each $s \in (0, \infty)$, the zero set of $u(s, \cdot)$ in $[-k, k]$, i.e. $Z_s = \{t \in [-k,k] \mid u(s,t) = 0\}$, is finite.

(2) The number of zeros of $u$ in $[-k, k]$ does not increase in $s$.

We will use this proposition in the following form:

**Proposition 4.2.8.** For any $v, w \in E_k$, the number of zeros of $\varphi_s(v) - \varphi_s(w)$ is non-increasing as $s$ increases.
Proof. Let \( u(s, t) := \varphi_s(v)(t) - \varphi_s(w)(t) \). Then because of (PDE), we have

\[
    u_s = u_{tt} + V_q(t, v) - V_q(t, w)
\]

(4.18)

\[
    = u_{tt} + \left( \int_0^1 V_{qq}(t, w + \alpha(v - w)) \, d\alpha \right) u
\]

(4.19)

Letting \( b(s, t) := \int_0^1 V_{qq}(t, w(s, t) + \alpha(v(s, t) - w(s, t))) \, d\alpha \), we apply the preceding proposition.

Notice that in particular, if \( q \) satisfies (HS), then \( q \) solves (PDE), and so the number of zeros of \( \varphi_s(u) - q \) does not increase in \( s \). The number of zeros of \( \varphi_s(u) - q \) tells how many times the graph of \( \varphi_s(u)(t) \) crosses the graph of \( q(t) \), or how many times \( \varphi_s(u)(t) \) intersects \( q(t) \). Thus, the non-increasing number of zeros means that the number of intersections of \( \varphi_s(u)(t) \) with \( q(t) \) does not increase in time. Moreover, since \( q \equiv 0, q \equiv 1 \) are solutions to (HS) and are invariant under \( \varphi_s \), note that if \( 0 \leq u \leq 1 \), then we must have \( 0 \leq \varphi_s(u) \leq 1 \) by the maximum principle. Thus, (DV6) is not necessary - we could change the potential \( V \) for \( x \notin [0, 1] \) so that (DV6) is satisfied.

We will also use the following refinement of Proposition 4.2.7, which is also contained in [1].

**Proposition 4.2.9.** If \( u \not\equiv 0 \) satisfies \( u_s = u_{tt} + b(s, t)u \) for \( t \in \mathbb{R} \), \( |u(s, t)| \) is bounded and \( b \in L^\infty(\mathbb{R}) \), then the zero set of \( u(s, \cdot) \) is discrete, and if this set is finite, then the number of zeros does not increase as \( s \) increases.

Next, we can find the solutions that we will use to find our homoclinic.

**Proposition 4.2.10.** There is a \( k_0 \in \mathbb{N} \) and a sequence of solutions \( q_k \in E_k \) to (HS) such that for \( k \geq k_0 \),
(1) \( a \leq I_k(q_k) \leq C \), where \( a, C \) are from Lemmas 4.2.3, 4.2.4.

(2) \( q_k = \lim_{s_i \to \infty} \varphi_{s_i}(q^A_k) \) for some sequence \( s_i \to \infty \), and \( q^A_k = h^k_r \) for some \( r \in [0, 1] \) (with \( r \) depending on \( k \)).

(3) If \( J_k \subset (-k, k) \) is the interval on which \( q^A_k(t) \neq 0 \), then the length of \( J_k \) is bounded independently of \( k \).

(4) \( q_k \) intersects \( 1/2 \) exactly twice in \([-k, k]\).

Proof. Since 0, 1 are solutions to (HS), and hence of (PDE), they are invariant under \( \varphi_s \). Because of the continuity of \( \varphi_s(u) \) in \( u \), for every \( s, r \mapsto \varphi_s(h^k_r) \), \( r \in [0, 1] \), is a path from 0 to 1 in \( E_k \). Thus, for every \( s \), there is a first \( r_s \in [0, 1] \) for which there is a \( t \) with \( \varphi_s(h^k_r_s)(t) = \frac{1}{2} \). We next show that \( r_s \) is increasing.

Suppose that \( s_1 < s_2 \). We will show that \( r_{s_1} \leq r_{s_2} \). It suffices to show that

\[
\{ r \in [0, 1] \mid \varphi_{s_1}(h^k_r) < 1/2 \} \subset \{ r \in [0, 1] \mid \varphi_{s_2}(h^k_r) < 1/2 \},
\]

since \( r_s = \sup \{ r \in [0, 1] \mid \varphi_s(h^k_r) < 1/2 \} \). Let \( \rho \in \{ r \in [0, 1] \mid \varphi_{s_1}(h^k_r) < 1/2 \} \). Then \( \varphi_{s_1}(h^k_{\rho}) < 1/2 \). But then by Lemma 4.2.8, \( \varphi_s(\varphi_{s_1}(h^k_{\rho})) < 1/2 \). Thus, \( \varphi_{s+s_1}(h^k_{\rho}) < 1/2 \) for all \( s \geq 0 \). Taking \( s = s_2 - s_1 \), we see that \( \varphi_{s_2}(h^k_{\rho}) < 1/2 \), and so (4.20) is satisfied.

Thus, as \( s \to \infty \), we have \( r_s \to r \). We now define \( q^A_k := h^k_r \). Notice that this means that the set \( J_k \) of points where \( q^A_k \neq 0 \) is indeed an interval, so (3) makes sense. (The correctness of the statement remains to be proven, but at least we now have some consistency.)

Invoking Theorem 4.2.5, there is a sequence \( s_i \to \infty \) as \( i \to \infty \) such that we may define \( q_k \) as in (2). Thus, \( q_k \) satisfies (HS). By Lemma 4.2.4, \( I_k(q^A_k) \leq C \),
and so by Lemma 4.2.6, we have $I_k(q_k) \leq C$. Let $r_i := r_{s_i}$. To show that $a \leq I_k(q_k)$, we proceed by contradiction. Suppose then that $I_k(q_k) < a$. Since $\varphi_{s_i}(q_k^A)$ converges to $q_k$ in $E_k$, we must then have $I_k(\varphi_{s_i}(q_k^A)) < a$ for all large $i$. Notice that $q_k^A = \lim_{j \to \infty} h^k_{r_j}$, hence for all large $j$ we have $I_k(\varphi_{s_i}(h^k_{r_j})) < a$. Moreover, we can take $j > i$. Now, letting $s$ increase from $s_i$ to $s_j$ and using Lemma 4.2.6 we have $I_k(\varphi_{s_j}(h^k_{r_j})) < a$. But recall that $r_j$ is chosen so that $\varphi_{s_j}(h^k_{r_j})$ just touches 1/2. Then Lemma 4.2.3 implies that $I_k(\varphi_{s_j}(h^k_{r_j})) \geq \alpha_k \geq a$, a contradiction. Thus, we must have $I_k(q_k) \geq a$, which proves (1).

To prove (4), we proceed as in the preceding paragraph. Notice that by construction, each $h^k$, crosses 1/2 at most twice, hence $q_k^A$ crosses 1/2 at most twice. By Proposition 4.2.8, we know that $q_k$ crosses 1/2 at most twice. Since $I_k(1/2) \to \infty$ as $k \to \infty$ and $I_k(q_k)$ is bounded, we know that $q \neq 1/2$. Thus, we cannot have $q_k$ intersecting 1/2 exactly once. Because of the periodicity of $q_k$, it then suffices to show that $q_k$ crosses 1/2 at least once. Suppose that this is false, so $q_k$ doesn’t cross 1/2 at all. Because $\varphi_{s_i}(q_k^A)$ converges to $q_k$ in $E_k$ as $i \to \infty$, we know that $\varphi_{s_i}(q_k^A)$ converges uniformly to $q_k$. Thus, for all large $i$, $\varphi_{s_i}(q_k^A)$ cannot intersect 1/2. Fix a large $i$. By the continuity of $\varphi_{s}(\cdot)$ on initial conditions and because $h^k_{r_j} \to h^k_r = q_k^A$ uniformly, we must also have $\varphi_{s_i}(h^k_{r_j})$ disjoint from 1/2. As above, take $j > i$, and let $s$ go from $s_i$ to $s_j$. By Proposition 4.2.8, we must have $\varphi_{s_j}(h^k_{r_j})$ disjoint from 1/2. But this contradicts the choice of $r_j$.

Finally, we need to prove (3). Since $I_k(q_k) \geq a$, Lemma 4.2.6 implies we must also have $I_k(q_k^A) \geq a$. Suppose then that the length of $J_k$ is unbounded as $k \to \infty$ along a subsequence. Notice that $q_k^A = h^k_r$, so (using the notation from Lemma
4.2.4), we have

\[
0 < a \leq I_k(q_k^A)
\]

\[
= \int_{J^p_\eta} \left( \frac{1}{2} (\dot{q}_k^A)^2 - V(t, q_k^A) \right) dt + \int_{J^p_\eta} \left( \frac{1}{2} (\dot{q}_k^A)^2 - V(t, \eta) \right) dt
\]

\[
+ \int_{J^{up}_\eta} \left( \frac{1}{2} (\dot{q}_k^A)^2 - V(t, q_k^A) \right) dt + \int_{J^{down}_\eta} \left( \frac{1}{2} (\dot{q}_k^A)^2 - V(t, q_k^A) \right) dt
\]

\[
= 0 + \int_{J^\eta} -V(t, \eta) dt + \int_{J^{up}_\eta} \left( \frac{1}{2} (\dot{q}_k^A)^2 - V(t, q_k^A) \right) dt
\]

\[
+ \int_{J^{down}_\eta} \left( \frac{1}{2} (\dot{q}_k^A)^2 - V(t, q_k^A) \right) dt
\]

\[
\leq -\frac{c_1}{2} |J^\eta_\eta| + M,
\]

where \( M \) is a bound for

\[
\int_{J^{up}_\eta} \left( \frac{1}{2} (\dot{q}_k^A)^2 - V(t, q_k^A) \right) dt + \int_{J^{down}_\eta} \left( \frac{1}{2} (\dot{q}_k^A)^2 - V(t, q_k^A) \right) dt
\]

That such a bound exists follows from the arguments of Lemma 4.2.4. Now, \( J_k = J^p_\eta \cup J^{up}_\eta \cup J^{down}_\eta \), and the sum of the lengths of the latter two intervals is at most 1. Thus, if \(|J_k|\) is unbounded, then we must have \(|J^p_\eta|\) is unbounded, so eventually (4.21) will be negative, which contradicts \( a > 0 \).

With this proposition in mind, note that on \([-k, k]\), each \( q_k \) intersects \( 1/2 \) exactly twice. Since \( q_k(t) \) is a solution of (HS), so too is \( q_k(t - i) \) for all \( i \in \mathbb{Z} \). Thus, after an appropriate translation, we may assume there is an interval \([a_k, b_k]\) with midpoint in \([0, 1)\) such that

\[
\frac{1}{2} < q_k(t) < 1 \text{ for } t \in (a_k, b_k)
\]

Let \( \Gamma_k := [a_k, b_k] \). Notice that we have \( b_k - a_k < 2k \) and \( 0 \leq (a_k + b_k)/2 < 1 \), hence \(-k \leq a_k \leq 0 < b_k < k + 1\). In addition, we know that \( \{t \in \mathbb{R} \mid q_k(t) > 1/2\} = \cup_{m \in \mathbb{Z}} (a_k + 2mk, b_k + 2mk) \).
Now, \( q_k(a_k) = q_k(b_k) = 1/2 \), and since \( q_k \neq 1/2 \), we must have \( \dot{q}_k(a_k) \neq 0 \) and \( \dot{q}_k(b_k) \neq 0 \). Moreover, we have \( q_k(t) < 1/2 \) for all \( t < a_k \) close to \( a_k \). Thus, \( q_k \) is increasing at \( a_k \), hence \( \dot{q}_k(a_k) > 0 \). Similarly, we must have \( \dot{q}_k(b_k) < 0 \). To be able to control the asymptotic behavior, we need to know more about these intervals \( \Gamma_k \). We will in fact show that there is a subsequence \( k_j \to \infty \) such that \( |\Gamma_{k_j}| \) is bounded. To do this, we make use of (4) of Proposition 4.2.10.

**Lemma 4.2.11.** If \( q \) is a periodic solution of \((HS)\), there must be a \( t^* \in [-k, k] \) such that \( V_q(t^*, q(t^*)) = 0 \).

*Proof.* Because of \((HS)\), it suffices to show that \( \ddot{q}(t) = 0 \) for some \( t \). This follows from the Mean Value Theorem: since \( q(-k) = q(k) \), there must be a point \( \hat{t} \in [-k, k] \) such that \( \dot{q}(\hat{t}) = 0 \). But (again by periodicity), there must be a point \( t^* \in [\hat{t}, \hat{t} + 2k] \) such that \( \ddot{q}(t^*) = 0 \). But then, shifting by \( 2k \) if necessary, we have \( t^* \in [-k, k] \). \( \square \)

**Lemma 4.2.12.** If \( q \in C^2(\mathbb{R}, \mathbb{R}) \) is a bounded solution to \((HS)\), there is a sequence \( \{t_n\} \) with \( t_n \to \infty \) such that \( V_q(t_n, q(t_n)) \to 0 \) as \( n \to \infty \). An analogous statement hold for \( t_n \to -\infty \).

As an immediate corollary, we have

**Corollary 4.2.13.** There are no solutions of \((HS)\) with \( 1/2 + \alpha < q(t) < 1 - \varepsilon \) for all \( t \), where \( \alpha, \varepsilon > 0 \).

*Proof.* of Lemma 4.2.12 As in the preceding lemma, it suffices to find a sequence with \( \ddot{q}(t_n) \to 0 \). Since \( q \) is bounded, for any sequence \( t_n \) with \( t_n \to \infty \), there is a \( c \in [0, 1] \) and a subsequence (which we relabel) such that \( q(t_n) \to c \) as \( n \to \infty \).
Without loss of generality, we may also assume $|t_{n+1} - t_n| \geq 1$. We now find a sequence with $\dot{q}(\bar{t}_n) \to 0$. By the mean value theorem, there is $\bar{t}_n \in [t_n, t_{n+1}]$ with

$$\dot{q}(\bar{t}_n) = \frac{q(t_{n+1}) - q(t_n)}{t_{n+1} - t_n},$$

(4.23)

and so $|\dot{q}(\bar{t}_n)| \leq |q(t_{n+1}) - q(t_n)|$. But this last term converges to 0 as $n \to \infty$ since both $q(t_{n+1}), q(t_n) \to c$. Thus, we have a sequence $\bar{t}_n \to \infty$ such that $\dot{q}(\bar{t}_n) \to 0$.

Repeating the argument with $q$ replaced by $\dot{q}$ yields a sequence $\tilde{t}_n \to \infty$ with $\ddot{q}(\tilde{t}_n) \to 0$.

In the next lemmas, we describe the behavior of the $q_k$ on “long” sub-intervals of $\Gamma_k$.

**Lemma 4.2.14.** Given any $\varepsilon, \delta > 0$, there is a $\Lambda = \Lambda(\varepsilon, \delta) > 0$ and $k_0$ such that if $k \geq k_0$, $[\bar{a}_k, \bar{b}_k] \subset \Gamma_k$ and $1/2 + \delta < q_k(t) < 1 - \varepsilon$ for all $t \in (\bar{a}_k, \bar{b}_k)$, then $\bar{b}_k - \bar{a}_k < \Lambda$.

**Proof.** Suppose this is false. Then, there exist $\varepsilon, \delta > 0$ for which there is a sequence $k_j \to \infty$ and intervals $[\bar{a}_{k_j}, \bar{b}_{k_j}]$ with $\bar{b}_{k_j} - \bar{a}_{k_j} \to \infty$ on which $1/2 + \delta < q_{k_j}(t) < 1 - \varepsilon$. By considering appropriate translations as we did in (4.22), we may assume that the midpoint of $[\bar{a}_{k_j}, \bar{b}_{k_j}] \in [0,1)$. Now, we may also assume that the intervals are nested, and $\mathbb{R} = \bigcup_j [\bar{a}_{k_j}, \bar{b}_{k_j}]$. Notice that $\|q_{k_j}\|_{L^\infty(\mathbb{R})}$ is bounded. Therefore, by (HS), $\|\ddot{q}_{k_j}\|_{L^\infty(\mathbb{R})}$ is also bounded. But then by Lemma 4.2.1, we know that $\dot{q}_{k_j}$ is also bounded in $L^\infty(\mathbb{R})$, hence $q_{k_j}$ is bounded in $C^2(\mathbb{R})$, and so passing to a subsequence that we relabel, we have $q_{k_j} \to q$ in $C^1_{loc}(\mathbb{R})$ as $k_j \to \infty$. Because of the nestedness of the intervals $[\bar{a}_{k_j}, \bar{b}_{k_j}]$, we know that on every such interval, we have $1/2 + \delta \leq q(t) \leq 1 - \varepsilon$, and so $1/2 + \delta \leq q(t) \leq 1 - \varepsilon$ for all $t \in \mathbb{R}$. Next, we show that $q$ satisfies (HS), which will contradict Lemma 4.2.12.
We claim that \( \{\ddot{q}_{k,j}\} \) is equicontinuous on \( \mathbb{R} \). Notice that from (HS), we have

\[
|\ddot{q}_{k,j}(t_2) - \ddot{q}_{k,j}(t_1)| = |(V_q(t_2, q_{k,j}(t_2)) - V_q(t_1, q_{k,j}(t_1)))|
\]

\[
\leq \left| \int_{t_1}^{t_2} \frac{d}{ds} \left(V_q(s, q_{k,j}(s)) \right) ds \right|
\]

\[
\leq \left| \int_{t_1}^{t_2} V_{qq}(s, q_{k,j}(s)) ds \right| + \left| \int_{t_1}^{t_2} V_{q}(s, q_{k,j}(s)) \dot{q}_{k,j}(s) ds \right|
\]

(4.24)

\[
\leq K|t_2 - t_1|
\]

since \( \|\dot{q}_{k,j}\|_{L^\infty(\mathbb{R})} \) is uniformly bounded by Lemma 4.2.1. Since \( \dot{q}_{k,j} \) is uniformly bounded, the Arzela-Ascoli theorem applies to \( \ddot{q}_{k,j} \), and we have \( q_{k,j} \) converges in \( C^2_{loc}(\mathbb{R}) \). But then, we must have \( \ddot{q}_{k,j}(t) \to \ddot{q}(t) \) and \( V_q(t, q_{k,j}(t)) \to V_q(t, q(t)) \), hence \( \ddot{q}(t) = V_q(t, q(t)) \).

\[\square\]

**Lemma 4.2.15.** For any \( \delta, \varepsilon > 0 \), there are at most two maximal intervals \([a, b] \subset \Gamma_k \) where \( q_k(t) \in (1/2 + \delta, 1 - \varepsilon) \)

**Proof.** Notice that by the definition of \( \Gamma_k \), we must have \( \dot{q}_k(a_k) > 0 \) and \( \dot{q}_k(b_k) < 0 \). By the intermediate value theorem, there must be \( t_k \in (a_k, b_k) \) such that \( \dot{q}_k(t_k) = 0 \).

There can be no points \( t \) between \( a_k \) and \( t_k \) at which \( \dot{q}_k(t) = 0 \), for if there was, then (since \( q_k \) is \( C^2 \)) by the mean value theorem, there is a \( \hat{t} \) between \( t \) and \( t_k \) where \( \ddot{q}_k(\hat{t}) = 0 \), and so \( q_k(\hat{t}) \in \{1/2, 1\} \). But this is impossible for \( \hat{t} \in (a_k, b_k) \), since on this interval, \( q_k \) lies strictly between \( 1/2 \) and \( 1 \). Thus, we must have \( \dot{q}_k(t) > 0 \) for all \( t \in (a_k, t_k) \). Similarly, we must have \( \dot{q}_k(t) < 0 \) for all \( t \in (t_k, b_k) \).

Therefore, there are at most two such intervals: one between \( a_k \) and \( t_k \) (where \( q_k \) is increasing), and one between \( t_k \) and \( b_k \) (where \( q_k \) is decreasing). \[\square\]
Notice that the two preceding lemmas imply that for any \( \varepsilon, \delta > 0 \), we have
\[
|\{ t \in \Gamma_k \mid 1/2 + \delta \leq q_k(t) \leq 1 - \varepsilon \}| < 2\Lambda(\varepsilon, \delta).
\]

**Lemma 4.2.16.** For any \( \varepsilon > 0 \), there is an \( M = M(\varepsilon) \) such that if \( |\Gamma_k| > M \), then there is a \( t^* \in \Gamma_k \) such that \( |q_k(t^*) - 1| < \varepsilon \).

**Proof.** If the lemma is false, then there is an \( \varepsilon > 0 \) and a sequence \( k_j \to \infty \) with \( |\Gamma_{k_j}| \to \infty \) and \( q_{k_j}(t) < 1 - \varepsilon \) for all \( t \in \Gamma_k \). Choose \( \delta > 0 \) such that for \( x \in [1/2, 1/2 + \delta] \), \( -V(t, x) \geq -\frac{c}{2} \). Now, for \( x \in [0, 1/2] \), we have \( -V(t, x) \geq 0 \) for all \( t \), and so
\[
C \geq \int_{-k_j}^{k_j} \left( \frac{1}{2} (\dot{q}_{k_j})^2 - V(t, q_{k_j}) \right) dt = \int_{a_{k_j}}^{a_{k_j} + 2k_j} \left( \frac{1}{2} (\dot{q}_{k_j})^2 - V(t, q_{k_j}) \right) dt \quad (4.25)
\]
\[
\geq \int \{ t \in \Gamma_{k_j} \mid q_{k_j}(t) > 1/2 + \delta \} -V(t, q_{k_j}) dt + \int \{ t \in \Gamma_{k_j} \mid 1/2 \leq q_{k_j} \leq 1/2 + \delta \} -V(t, q_{k_j}) dt \quad (4.26)
\]
But \( |\{ t \in \Gamma_{k_j} \mid q_{k_j}(t) > 1/2 + \delta \}| \) is bounded independently of \( k_j \) by Lemma 4.2.14 and 4.2.15, since on \( \Gamma_{k_j} \), \( q_{k_j}(t) < 1 - \varepsilon \). Thus, the first term is bounded, and so we must have \( |\{ t \in \Gamma_{k_j} \mid 1/2 \leq q_{k_j} \leq 1/2 + \delta \}| \to \infty \). Because \( -V(t, x) \geq -\frac{c}{2} \) for \( x \in [1/2, 1/2 + \delta] \), the second term must go to \( \infty \), which is impossible. \( \square \)

Finally, we can prove the following.

**Proposition 4.2.17.** There is a subsequence \( k_j \to \infty \) such that \( |\Gamma_{k_j}| = b_{k_j} - a_{k_j} \) is bounded.

**Proof.** Arguing indirectly, suppose that \( |\Gamma_k| \) is unbounded. Thus, there is a sequence \( k_j \to \infty \) such that \( |\Gamma_{k_j}| < |\Gamma_{k_{j+1}}| \to \infty \). By Lemma 4.2.16, we then know that for every \( \varepsilon > 0 \), there is a \( J = J(\varepsilon) \) such that for all \( j > J \), there is a \( t_j \in \Gamma_{k_j} \) with \( q_{k_j}(t_j) > 1 - \varepsilon \). Let \( \hat{t}_j \in \Gamma_{k_j} \) be the point at which \( 1 - q_{k_j}(t) \) is smallest,
i.e. \( \hat{t}_j \) is the point in \( \Gamma_{k_j} \) where \( q_{k_j}(t) \) is closest to 1. In particular, \( \dot{q}_{k_j}(\hat{t}_j) = 0 \) and \( q_{k_j}(\hat{t}_j) > 1 - \varepsilon \) for \( j > J \). By the continuous dependence of solutions to (HS) on their initial data, there will be an interval around \( \hat{t}_j \) where \( q_{k_j}(t) \) is close to 1. This is because \( q_{k_j}(\hat{t}_j) \) is close to 1, and \( \dot{q}_{k_j}(\hat{t}_j) = 0 \), and the solution \( u \) to (HS) satisfying \( u(\hat{t}_j) = 1, \dot{u}(\hat{t}_j) = 0 \) is \( u \equiv 1 \). More precisely, for any given \( L, \lambda \), there is an \( \varepsilon(L, \lambda) \) such that \( q_{k_j}(t) > 1 - \lambda \) for all \( t \in [\hat{t}_j - L, \hat{t}_j + L] \), \( j > J(\varepsilon(L, \lambda)) \).

Now, by (3) of Proposition 4.2.10 there is an \( M > 0 \) such that if \( J_k \subset [-k, k] \) is the interval centered at 0 where \( q_A^{k} \neq 0 \), then \( |J_k| \leq M \). Moreover, we know that \( q_A^{k}(t) \leq \eta \) for all \( t \). We next show that there is a \( j \in \mathbb{N} \) and a \( p \in \mathbb{Z} \) such that \( q_A^{k_j}(t) < \tau_p q_{k_j}(t) \) (4.27) for all \( t \in [-k_j, k_j] \).

Pick \( \lambda \) small enough that \( 1 - \lambda > \eta \) (where \( \eta \) is from the definition of the homotopy), and let \( L = M + 1 \). Then, pick \( p \in \mathbb{Z} \) such that \( \hat{t}_j + p \in [0, 1) \). Pick a \( j > J \). We claim that \( q_A^{k_j}(t) < \tau_p q_{k_j}(t) \) for all \( t \in \mathbb{R} \). Since \( q_{k_j}(t) > 0 \), it suffices to show the inequality above on the set of points where \( q_A^{k}(t) \neq 0 \). In fact, we shall show that \( \tau_p q_{k_j}(t) > 1 - \lambda > \eta \geq q_A^{k}(t) \) for \( t \in J_{k_j} \) (where \( J_{k_j} \) is the interval centered at 0 where \( q_A^{k_j}(t) \neq 0 \)). Notice that \( J_{k_j} = [-\xi, \xi] \) for some \( \xi \leq M/2 \).

Thus, \( t \in J_{k_j} \) implies \( -M/2 \leq t \leq M/2 \). Now, \( M < \hat{t}_j + p + L \). In addition, \( \hat{t}_j + p - L < 1 - L = -M < -M/2 \), hence \( -M/2 < t < M/2 \) implies \( \hat{t}_j + p - L < t < \hat{t}_j + p + L \). But in this last interval, \( \tau_p q_{k_j}(t) = q_{k_j}(t - p) > 1 - \lambda > \eta \geq q_A^{k_j}(t) \), and so (4.2) holds.

In fact, we can take \( p = 0 \). To see this, note that by (4.2) and the fact that \( \tau_p q_{k_j} \) solves (HS) and is invariant under \( \varphi_s \), we must have \( \tau_p q_{k_j}(t) > \varphi_s(q_A^{k_j})(t) \) for
all \( s_i \to \infty \). Thus, \( \tau_p q_{k_j}(t) \geq q_{k_j}(t) \). We cannot have \( \tau_p q_{k_j}(t) > q_{k_j}(t) \) for all \( t \), since the \( q_{k_j} \) are periodic and nonconstant. If we have equality at any point, then by the uniqueness of solutions to ordinary differential equations, we must have \( \tau_p q_{k_j} \equiv q_{k_j} \), and so \( p = 2mk_j \) for some \( m \in \mathbb{Z} \). In particular, we can take \( p = 0 \).

Now, we fix \( k_j \). Consider the sequence \( l k_j \) for \( l \to \infty \). Because of the periodicity of \( q_{k_j} \), \( q_{k_j} \in E_{l k_j} \) for all \( l \in \mathbb{Z} \). From (3) of Proposition 4.2.10, \( |J_{k_j}| \leq M \) for all \( l \). Thus, we can replace \( J_{k_j} \) with \( J_{l k_j} \) in the argument above to get \( q_{k_j}(t) > q_{l k_j}^A(t) \) for \( t \in [-l k_j, l k_j] \) and all \( l \in \mathbb{Z} \). By Proposition 4.2.8, the number of intersections of \( \varphi_s(q_{l k_j}^A) \) with \( q_{k_j} \) is non-increasing as \( s \to \infty \). Because \( q_{l k_j}^A \) is disjoint from \( q_{k_j} \), we then know that \( q_{l k_j} \leq q_{k_j} \). In fact, for all \( l \in \mathbb{N} \), \( l \geq 2 \), we must have \( q_{l k_j} < q_{k_j} \), since both solve (HS) and they are not equal.

Recall that by definition, \( \Gamma_{l k_j} = [a_{l k_j}, b_{l k_j}] \), where \( (a_{l k_j}, b_{l k_j}) \) is an interval with midpoint in \([0,1)\) and on which \( q_{l k_j}(t) > 1/2 \). Since \( q_{k_j} > q_{l k_j} \), we know that \( 1/2 \leq q_{l k_j}(t) < q_{k_j}(t) \) for all \( t \in \Gamma_{l k_j} \). Since \( \Gamma_{l k_j} \) is connected, and the maximal intervals on which \( q_{k_j}(t) > 1/2 \) are of the form \( (a_{k_j} + 2mk_j, b_{k_j} + 2mk_j) \), we know that \( (a_{l k_j}, b_{l k_j}) \subset (a_{k_j} + 2mk_j, b_{k_j} + 2mk_j) \) for some \( m \in \mathbb{Z} \). But then \( b_{l k_j} - a_{l k_j} \leq b_{k_j} - a_{k_j} \), and so the sequence \( c_{l k_j}, l \to \infty \), satisfies the proposition. \( \square \)

Next, we prove a lemma that we use to show that the limit of the appropriate subharmonics has the correct asymptotics.

**Lemma 4.2.18.** Suppose that we have a sequence \( v_j \in E_{k_j} \) of solutions of (HS) such that

(i) \( 0 \leq v_j(t) \leq 1 \) for all \( j \)

(ii) \( I_{k_j}(v_j) \leq C \)
(iii) There is an $M$ such that \( \{ t \in [-k_j, k_j] \mid v_j(t) > 1/2 \} \subset [-M, M] \) for all sufficiently large $j$.

Then there is a solution $w$ of (HS) which is the local uniform limit of the $v_j$ such that $w(t), \dot{w}(t) \to 0$ as $|t| \to \infty$.

Proof. Since $0 \leq v_j(t) \leq 1$ for all $t$ and $\ddot{v}_j = -V_q(t, v_j(t))$, we know that $\|v_j\|_{L^\infty(\mathbb{R})}$ and $\|\ddot{v}_j\|_{L^\infty(\mathbb{R})}$ are bounded. But then, by Lemma 4.2.1, we know that $\dot{v}_j$ is bounded in $L^\infty(\mathbb{R})$, and so by the Arzela-Ascoli theorem, there is a $w \in C^1(\mathbb{R})$ such that by passing to a subsequence (which we relabel) $v_j \to w$ in $C^1_{loc}(\mathbb{R})$. Repeating the argument in Lemma 4.2.14, we see that $w$ solves (HS).

We now turn to the most interesting question: the asymptotic behavior of $w$. Suppose that $w(t) \not\to 0$ as $t \to \infty$. Then, there must be an $\varepsilon > 0$ and a sequence $t_n \to \infty$ such that $w(t_n) > \varepsilon$ for all $n$. Because of the uniform convergence of $v_j$ to $w$ on bounded intervals, for any $m \in \mathbb{N}$ and any $\delta > 0$, there is a $j_1 = j_1(m, \delta)$ such that if $j \geq j_1$, then $|v_j(t_i) - w(t_i)| < \delta$ for $i = 1, 2, \ldots, m$. Picking $\delta$ appropriately small, we will have $v_j(t_i) \geq \varepsilon/2$ for $i = 1, 2, \ldots, m$.

We now use this bound on $\|\dot{v}_j\|_{L^\infty(\mathbb{R})}$ to construct intervals of uniform length, centered at $t_i$ and on which $v_j(t) \geq \varepsilon/4$. We have

\[
|v_j(t) - v_j(t_i)| = \left| \int_{t_i}^{t} \dot{v}_j(s) ds \right| \leq |t - t_i| \|\dot{v}_j\|_{L^\infty(\mathbb{R})} \leq M|t - t_i|
\]

In particular, we have

\[
\frac{\varepsilon}{2} < v_j(t_i) \leq |v_j(t) - v_j(t_i)| + v_j(t) \quad (4.29)
\]
since $v_j \geq 0$, and so

$$\frac{\varepsilon}{2} - M|t - t_i| < v_j(t). \quad (4.30)$$

Thus, if $|t - t_i| \leq \left(\frac{\varepsilon}{4M}\right)$, we will have

$$\frac{\varepsilon}{4} \leq \frac{\varepsilon}{2} - M|t - t_i| < v_j(t). \quad (4.31)$$

More plainly stated, there is an interval of length $2\left(\frac{\varepsilon}{4M}\right)$ centered at $t_i$ on which we have $v_j(t) \geq \frac{\varepsilon}{4}$.

Next, by (DV5), there is a $\Lambda$ such that $-V(t, x) \geq \Lambda x^2$ for $0 \leq x \leq 1/2$. Thus, we have

$$C \geq \int_{-k_j}^{k_j} \left(\frac{1}{2}(\dot{v}_j)^2 - V(t, v_j)\right) dt$$

$$\geq \int_{1/2 \leq v_j(t) \leq 1} -V(t, v_j) \, dt + \int_{0 \leq v_j(t) \leq 1/2} -V(t, v_j) \, dt$$

$$\geq \Lambda \left(\int_{0 \leq v_j(t) \leq 1/2} (v_j)^2 \, dt\right) + \int_{1/2 \leq v_j(t) \leq 1} -V(t, v_j) \, dt$$

$$\geq \Lambda \sum_{i=1}^{m} \left(\int_{\text{interval around } t_i} (v_j)^2 \, dt\right) + \int_{1/2 \leq v_j(t) \leq 1} -V(t, v_j) \, dt$$

$$\geq \left(\text{const}\right) \sum_{i=1}^{m} \varepsilon^2 + \left(\int_{1/2 \leq v_j(t) \leq 1} -V(t, v_j) \, dt\right)$$

Now, we claim that (iii) implies that $\left(\int_{1/2 \leq v_j(t) \leq 1} -V(t, v_j) \, dt\right)$ is bounded independently of all large $j$. If this is true, then by taking $m$ to be suitably large, the right side in (4.32) will arbitrarily large, and we will have a contradiction to (ii), so we must in fact have $w(t) \to 0$ as $t \to \infty$. A similar argument shows that $w(t) \to 0$ as $t \to -\infty$. 
To verify our claim, we have for all sufficiently large $j$ that

\[
\left| \int_{1/2 \leq v_j(t) \leq 1} -V(t, v_j) \, dt \right| \leq \left( \max_{t \in [0,1], x \in [1/2,1]} |V(t, x)| \right) \left| \{ t \in [-k_j, k_j] \mid v_j(t) > 1/2 \} \right|
\]

(4.33)

\[
\leq 2M \max_{t \in [0,1], x \in [1/2,1]} |V(t, x)|
\]

by (iii).

Finally, we show that $\dot{w}(t) \to \infty$. By (HS), $\ddot{w}(t) = -V_q(t, w(t))$, and $w(t) \to 0$ as $|t| \to \infty$. Since $V_q(t,0) \equiv 0$, we must have $\ddot{w}(t) \to 0$ as $|t| \to \infty$. But then Lemma 4.2.2 shows us that $\dot{w}(t) \to 0$ as $|t| \to \infty$.

Now, we can prove the goal of this section:

**Theorem 4.2.19.** If $V$ satisfies (DV1-7), then there is a solution $0 \neq q$ to (HS) such that there are exactly two points $a, b$ where $q(a) = 1/2 = q(b)$ and $q(t), \dot{q}(t) \to 0$ as $t \to \pm \infty$.

**Proof.** Let $q_{k_j}$ be a sequence given by Proposition 4.2.17. We wish to apply Lemma 4.2.18. Clearly, we have (i), and (ii) follows from Proposition 4.2.10 with $v_j = q_{k_j}$. Recall that $\Gamma_{k_j}$ is chosen so that its midpoint lies in $[0,1)$. We also have $|\Gamma_{k_j}|$ is bounded by Proposition 4.2.17. Thus, there is an $M$ such that $|\Gamma_{k_j}| < M$. Thus, $b_{k_j} - a_{k_j} < M$, hence $b_{k_j} < M + a_{k_j} \leq M + 1$ since $a_{k_j} \leq 1$. Similarly, $-a_{k_j} < M - b_{k_j} < M$ since $b_{k_j} > 0$. But then $-M < a_{k_j}$. Thus, we have $-M - 1 < a_{k_j} < b_{k_j} < M + 1$, so $\Gamma_{k_j} \subset [-M - 1, M + 1]$. For $J$ sufficiently large that $j > J$ implies that $k_j > M + 1$, we will have $\Gamma_{k_j} \subset [-k_j, k_j]$, and thus $\Gamma_{k_j} = \{ t \in [-k_j, k_j] \mid q_{k_j}(t) > 1/2 \} \subset [-M - 1, M + 1]$, which gives us (iii). Thus,
Lemma 4.2.18 implies that there is a solution $q$ of (HS) such that $q(t), \dot{q}(t) \to 0$ as $|t| \to \infty$. It remains only to show that $q$ intersects $1/2$ exactly twice.

Because the midpoint of the interval where $q_{kj}(t) \in (1/2, 1)$ is in $[0, 1)$, there must be a $\hat{t} \in [0, 1]$ such that $q(\hat{t}) \geq 1/2$, hence $q \not\equiv 0$. (Note that Lemma 4.2.18 does not rule out the possibility that $q \equiv 0$.) In fact, we must have $q(\hat{t}) > 1/2$, since $q \not\equiv 1/2$ by the fact that $q(t) \to 0$ as $|t| \to \infty$. Therefore, $q$ crosses $1/2$ at least twice.

Notice that since $\Gamma_{kj} \subset [-M - 1, M + 1]$ for all large $j$, the only places in $[-k_j, k_j]$ where $q_{kj}(t) = 1/2$ are in $[-M - 1, M + 1]$. But then, for any $t \in [-k_j, k_j] \setminus [-M - 1, M + 1]$, we must have $q_{kj}(t) < 1/2$ for all $k_j$. Suppose then that $t^* \not\in [-M - 1, M + 1]$. For all sufficiently large $j$, we have $t^* \in [-k_j, k - j] \setminus [-M - 1, M + 1]$, hence $q_{kj}(t^*) < 1/2$ for all large $j$. But then $q(t^*) \leq 1/2$. Since $q \not\equiv 1/2$, we must have $q(t^*) < 1/2$. Thus, if $q(t)$ crosses $1/2$, it must happen in $[-M - 1, M + 1]$, and so $\{t \mid q(t) > 1/2\} \subset [-M - 1, M + 1]$.

Suppose now that $q$ crosses $1/2$ more than twice. Since $q$ must cross $1/2$ an even number of times, there are at least two maximal disjoint intervals $(a, b)$ and $(c, d)$ where $q(t) > 1/2$. Let $t_1$ be the point in $(a, b)$ where $q$ is closest to 1, and similarly let $t_2$ be the point in $(c, d)$ where $q$ is closest to 1. Note that there is a $t_3 \in (b, c)$ such that $q(t_3) < 1/2$. Because the $q_{kj}$ converge to $q$ uniformly in $[-M, M]$, for all large $j$, we must have $q_{kj}(t_1) > 1/2$, $q_{kj}(t_2) > 1/2$ and $q_{kj}(t_3) < 1/2$. Since $q_{kj}(-M) < 1/2$ and $q_{kj}(M) < 1/2$, $q_{kj}$ crosses $1/2$ at least twice between $-M$ and $t_3$, and at least twice between $t_3$ and $M$. Thus, we must have $q_{kj}$ crossing $1/2$ at least four times in $[-M, M]$, which is impossible. Thus, $q$ must cross $1/2$ at most twice. \[\square\]
Remark 4.2.20. Notice that the proof above relies on the following dichotomy:

(i) Either for all $k$ the amount of time for which the $q_k$ are larger than $1/2$ is bounded independently of $k$, or

(ii) there is a subsequence $k_j \to \infty$ as $j \to \infty$ for which the amount of time where $q_{k_j}(t)$ is larger than $1/2$ tends to $\infty$ as $j \to \infty$.

In case (i), it is fairly easy to find a homoclinic solution: By considering appropriate translations, we may assume that there is an $M$ such that for $|t| > M$, $q_k(t) < 1/2$ for all $k$. Since the $q_k$ are bounded in $C^2(\mathbb{R})$, and $\ddot{q}_k$ is equicontinuous, the $q_k$ converge in $C^2_{loc}(\mathbb{R})$ to some $q$, a solution of (HS). For $|t| > M$, we must have $q(t) < 1/2$, and then Lemma 4.2.18 gives us a homoclinic solution. In case (ii), we showed that there is a large $k^*$ such that for all $l \geq 1, l \in \mathbb{N}$ we have $q_{k^*}(t) > q_{l^*k^*}(t)$, and so $q_{k^*}(t) > q_{l^*k^*}(t)$. Then we can use the argument of case (i) on the subsequence $q_{l^*k^*}$.

An interesting question is to determine when these two cases can occur, especially since as will see in Section 3 case (ii) implies the existence of very interesting dynamics. With this in mind, let us replace (DV5) by

(DV5)' The only zeros of $V_q(t, x)$ are $x = 0, 1/2, 1$.

Notice that (DV5)' implies (DV5), and so all of our results so far apply to this case. Next, we discuss the behavior of the solutions $q_k$ under the assumption (DV5)'. Since $\ddot{q}_k(t) = -V_q(t, q_k(t))$, by (DV5)', we know that for $t \in (a_k, b_k)$, we have $\ddot{q}_k(t) < 0$, hence $\dot{q}_k$ is strictly decreasing in $(a_k, b_k)$. Let $\tilde{t}_k$ be the point in $(a_k, b_k)$ where $q_{k_j}(t)$ is closest to 1. Similarly, (DV5)' implies that $\ddot{q}_k(t) > 0$ for
Thus, on this interval, \( \dot{q}_k(t) \) is strictly increasing on \((b_k, a_k + 2k)\). Let \( t_k \) be the point in \((b_k, a_k + 2k)\) where \( q_k(t) \) is closest to 0. This discussion proves the following:

**Lemma 4.2.21.**  
(i) For \( t \in (a_k, \bar{t}_k) \), \( \dot{q}_k(t) > 0 \), so \( q_k \) is strictly increasing on \((a_k, \bar{t}_k)\).

(ii) For \( t \in (\bar{t}_k, b_k) \), \( \dot{q}_k(t) < 0 \), so \( q_k \) is strictly decreasing on \((\bar{t}_k, b_k)\).

(iii) For \( t \in (b_k, t_k) \), \( \dot{q}_k(t) < 0 \), so \( q_k \) is strictly decreasing on \((b_k, t_k)\).

(iv) For \( t \in (t_k, a_k + 2k) \), \( \dot{q}_k(t) > 0 \), so \( q_k \) is strictly increasing on \((t_k, a_k + 2k)\).

We can summarize the previous lemma in the following diagram:

![Graph showing the behavior of \( q_k \) around the critical points \( a_k \) and \( b_k \).]

We now show that if \( V \) satisfies (DV1-4), (DV5)', and (DV6-7), then we must have case (i). With this in mind, for the remainder of this section, suppose that \( V \) satisfies (DV1-4), (DV5)' and (DV6-7).

We start with a straightforward extension of Lemma 4.2.12.
Lemma 4.2.22. If \( q \in C^2(\mathbb{R}, \mathbb{R}) \) is a bounded solution to (HS), there is a sequence \( \{t_n\} \) with \( t_n \to \infty \) such that \( q(t_n) \to \psi \) for \( \psi \in \{0, \frac{1}{2}, 1\} \). An analogous statement hold for \( t_n \to -\infty \).

Proof. We may simply copy the proof of Lemma 4.2.12, noting that since \( V_q(t, x) = 0 \) if and only if \( x = 0, 1/2, 1 \), the sequence \( q(t_n) \to \psi \) for \( \psi \in \{0, 1/2, 1\} \).

As for Lemma 4.2.12, we have the following

Corollary 4.2.23. There is no solution of \( v_k \in E_k \) of (HS) such that \( 0 < v_k(t) < 1/2 \) or \( 1/2 < v_k(t) < 1 \) for all \( t \in \mathbb{R} \).

The next lemma tells us that if we have (DV5)' instead of (DV5), then the \( q_k \) are minimal, in the sense that there are no periodic solutions lying strictly between 0 and \( q_k \).

Lemma 4.2.24. The \( q_k \) found in Proposition 4.2.10 under hypothesis (DV5)' are minimal. That is, there is no solution \( w \in E_k \) of (HS) such that \( 0 < w < q_k \).

Proof. Suppose that there is such a \( w \) for some \( k^* \in \mathbb{N} \). Then, for some large \( i \), we must have \( \varphi_{s_k}(q_k^{k^*}) > w \). But then for all large \( j \), we must have \( \varphi_{s_k}(h_{r_j}^{k^*}) > w \). Without loss of generality, we can take some \( j > i \). Then, by Proposition 4.2.8, we will have

\[
w < \varphi_{s_{j-s_k}}(\varphi_{s_k}(h_{r_j}^{k^*})) = \varphi_{s_j}(h_{r_j}^{k^*}) \leq 1/2.
\]

But by Corollary 4.2.23, this implies that \( w \equiv 0 \), which contradicts our assumption about \( w \).

Our next lemma tells us 0 is an isolated solution of (HS), and 0 is an asymptotically stable fixed point of the flow \( \varphi_s \).
Lemma 4.2.25. There is an $\varepsilon_1 > 0$ such that if $0 \leq w < \varepsilon_1$, then $\|\varphi_s(w)\|_{E_k} \to 0$ as $s \to \infty$.

Proof. Since $0$ is a local maximum of $V$, and $V(t, 0) \equiv 0$, there is an $\varepsilon_1 \in (0, 1/2)$ such that for $x \in (0, \varepsilon_1]$, $V_q(t, x) < 0$. Let $u \equiv \varepsilon_1$. Then $u_{tt} + V_q(t, u) < 0$. Notice that (letting $w(s, t) := \varphi_s(u)(t)$) we have $w_s(0, t) = u_{tt}(0, t) + V_q(t, u(0, t)) < 0$, and $(w_s)_s = (w_s)_{tt} + V_{qq}(t, w)w_s$. Thus, $w_s$ satisfies a linear parabolic PDE in $[-k, k]$. Since the initial condition for $w_s$ is less than zero and $0$ is a solution of $(w_s)_s = (w_s)_{tt} + V_{qq}(t, w)(w_s)$, the maximum principle for linear parabolic PDEs tells us that we must have $w_s(s, t) < 0$ for all $s > 0$. But this means that for each fixed $t$, $s \mapsto w(s, t)$ is decreasing in $s$, and so if $s_1 < s_2$, then $\varphi_{s_1}(u) > \varphi_{s_2}(u)$. By Lemma 4.2.22, there are no periodic solutions of (HS) that lie strictly between 0 and 1/2, and so $\varphi_s(u)$ must converge to 0 in $E_k$ as $s \to \infty$.

Suppose now that $0 \leq w < \varepsilon_1$. We must have $0 \leq \varphi_s(w) < \varphi_s(\varepsilon_1)$ for all $s \geq 0$, because the number of crossings of two solutions of the parabolic flow is non-increasing. If $\varphi_s(w) \not\to 0$, then there is a sequence $s_i \to \infty$ such that $\varphi_{s_i}(w) \to v \geq 0$ for some $v$ a solution to (HS). In face, we must have $v > 0$, because if $v(t) = 0$ for some $t$, then $v \equiv 0$. But this is impossible, since $0 \leq \varphi_{s_i}(w) < \varphi_{s_i}(\varepsilon_1) \to 0$ in $E_k$, and so in $L^\infty$. \qed

Next, we prove an analog of the lemma above for $q_k$ give by Proposition 4.2.10.

Lemma 4.2.26. Suppose there is a $k \in \mathbb{N}$ such that $q_k^A < q_k$ and $\varphi_{s_i}(q_k^A) \to q_k$ as $i \to \infty$. Then for any $w \in E_k$ with $q_k^A < w \leq q_k$, we have $\varphi_s(w) \to q_k$ as $s \to \infty$.

Notice that while we only assume that $\varphi_{s_i}(q_k^A) \to \infty$ as $i \to \infty$, we conclude that $\varphi_s(w) \to q_k$ for $s \to \infty$. Notice that we cannot assume that $\varphi_s(q_k^A) \to q_k$
as \( s \to \infty \), as Theorem 4.2.5 only guarantees convergence on subsequences. If we knew that \( q_k \) were isolated in \( E_k \), we would indeed have \( \varphi_s(q_k^A) \to q_k \).

**Proof.** (of Lemma 4.2.26) Suppose that the statement is false. Then there is a \( w \in E_k \) with \( q_k^A < w \leq q_k \) and a sequence \( \tilde{s}_i \to \infty \) such that \( \varphi_{\tilde{s}_i}(w) \not\to q_k \).

By passing to a subsequence again, we may assume that \( \varphi_{\tilde{s}_i}(w) \to \tilde{q} \). Notice that we must have \( 0 \leq \tilde{q} \leq q_k \). By Lemma 4.2.24 and the assumption, we must then have \( \tilde{q} \equiv 0 \). Notice that this means that \( \varphi_{\tilde{s}_i}(w) \to 0 \) in \( E_k \). Thus, for some large \( i \) we have \( 0 \leq \varphi_{\tilde{s}_i}(w) < \varepsilon_1 \), hence \( 0 \leq \varphi_{\tilde{s}_i}(q_k^A) < \varphi_{\tilde{s}_i}(w) < \varepsilon_1 \) and so by Lemma 4.2.25, we know that \( \varphi_s(\varphi_{\tilde{s}_i}(q_k^A)) \to 0 \) in \( E_k \). Now, for \( l \) large, we will have \( s_l > \tilde{s}_i \), so taking \( s := s_l - \tilde{s}_i > 0 \) and letting \( l \to \infty \) we must have \( \varphi_{s_l}(q_k^A) = \varphi_{s_l-\tilde{s}_i}(\varphi_{\tilde{s}_i}(q_k^A)) \to 0 \), which is impossible. \( \square \)

Next, we prove a simple version of a Theorem of Matano ([14]). In essence, it says that between any two stable solutions, there is a third solution.

**Proposition 4.2.27.** Let \( v_1 < v_2 \) be two solutions of (HS) in \( E_k \), and suppose there exists \( \varepsilon_1, \varepsilon_2 > 0 \) such that for any \( w \in E_k \) with \( v_1 \leq w < v_1 + \varepsilon_1, \varphi_s(w) \to v_1 \) as \( s \to \infty \), and for any \( w \in E_k \) with \( v_2 - \varepsilon_2 < w \leq v_2, \varphi_s(w) \to v_2 \) as \( s \to \infty \). Then there is a solution \( v_3 \in E_k \) of (HS) such that \( v_1 < v_3 < v_2 \).

**Proof.** Consider the homotopy \( h_r(t) := rv_2(t) + (1-r)v_1(t) \). Notice that if \( r_1 < r_2 \), then \( h_{r_1} < h_{r_2} \). Since \( v_i \) is a solution of (HS), \( \varphi_s \) fixes \( v_i, i = 1, 2 \). Thus, we know that for each \( j \in \mathbb{N}, r \mapsto \varphi_j(h_r) \) is a homotopy from \( v_1 \) to \( v_2 \). Recalling that the number of intersections is non-increasing, if \( r_1 < r_2 \) then \( \varphi_j(h_{r_1}) < \varphi_j(h_{r_2}) \).

Now, for each \( j \), notice that for all \( r \) sufficiently close to 0 (or 1), we must have
\( v_1 \leq \varphi_j(h_r) < v_1 + \varepsilon_1 \) (or \( v_2 - \varepsilon < \varphi_j(h_r) \leq v_2 \)). Hence

\[
\{ r \in [0, 1] \mid \varphi_s(\varphi_j(h_r)) \to v_1 \text{ as } s \to \infty \} \neq \emptyset \quad (4.34)
\]

and

\[
\{ r \in [0, 1] \mid \varphi_s(\varphi_j(h_r)) \to v_2 \text{ as } s \to \infty \} \neq \emptyset. \quad (4.35)
\]

Next, suppose that \( \varphi_s(\varphi_j(h_{r_1})) \to v_1 \) as \( s \to \infty \). Then, for some large \( \bar{s} \), we must have \( v_1 \leq \varphi_{\bar{s}}(\varphi_j(h_{r_1})) < v_1 + \varepsilon \). But then for all \( r \) close to \( r_1 \), we must still have \( v_1 \leq \varphi_{\bar{s}}(\varphi_j(h_r)) < v_1 + \varepsilon_1 \), and so \( \varphi_s(\varphi_{\bar{s}}(\varphi_j(h_r))) \to v_1 \) as \( s \to \infty \). Let \( \hat{s} := s - \bar{s} > 0 \) for all large \( s \). We must then have \( \varphi_s(\varphi_j(h_r)) \to v_1 \) as \( \hat{s} \to \infty \).

Thus, \( \{ r \in [0, 1] \mid \varphi_s(\varphi_j(h_r)) \to v_1 \text{ as } s \to \infty \} \) is an open set. Next, notice that since \( \varphi_j(h_{r_1}) < \varphi_j(h_{r_2}) \) for \( r_1 < r_2 \), and \( \varphi_s \) preserves order, we know that if \( \varphi_s(\varphi_j(h_r)) \to v_1 \text{ as } s \to \infty \), then \( \varphi_s(\varphi_j(h_r)) \to v_1 \text{ as } s \to \infty \) for all \( r < r^* \).

Therefore, there exists and \( \xi_{j_0} > 0 \) such that

\[
\{ r \in [0, 1] \mid \varphi_s(\varphi_j(h_r)) \to v_1 \text{ as } s \to \infty \} = [0, \xi_{j_0}). \quad (4.36)
\]

Similarly, there must be \( \bar{r}_j < 1 \) such that

\[
\{ r \in [0, 1] \mid \varphi_s(\varphi_j(h_r)) \to v_2 \text{ as } s \to \infty \} = (\bar{r}_j, 1]. \quad (4.37)
\]

Notice that we must also have \( \xi_{j_0} \leq \bar{r}_j \).

Next, we show that

\[
\xi_{j_0} \leq \xi_{j_0 + 1}. \quad (4.38)
\]

It suffices to show that

\[
\{ r \in [0, 1] \mid \varphi_s(\varphi_j(h_r)) \to v_1 \text{ as } s \to \infty \}
\subset \{ r \in [0, 1] \mid \varphi_s(\varphi_{j+1}(h_r)) \to v_1 \text{ as } s \to \infty \}.
\]
Suppose then that \( \varphi_s(\phi_j(h_\rho)) \to v_1 \) as \( s \to \infty \). Then clearly \( \varphi_s(\phi_{j+1}(h_\rho)) = \varphi_{s+1}(\phi_j(h_\rho)) \to v_1 \) as \( s \to \infty \), hence \( \rho \in \{ r \in [0,1] \mid \varphi_s(\phi_{j+1}(h_r)) \to v_1 \text{ as } s \to \infty \} \).

A similar argument shows that

\[
\bar{r}_{j+1} \leq \bar{r}_j. \tag{4.39}
\]

Notice that this means that \( \underline{r}_j \to r \) and \( \bar{r}_j \to \bar{r} \) as \( j \to \infty \), and \( r \leq \bar{r} \).

Let \( r^* \in [\underline{r}, \bar{r}] \). By Theorem 4.2.5, there is a solution \( w \) of (HS) in \( E_k \) such that \( \varphi_{s_i}(h_{r^*}) \to w \) in \( E_k \) as \( s_i \to \infty \). Since \( v_1 \leq h_r \leq v_2 \), we must have \( v_1 \leq w \leq v_2 \).

Suppose now that \( \varphi_{s_i}(h_{r^*}) \to v_1 \) as \( s_i \to \infty \). Then, for some large \( i \), we have \( v_1 \leq \varphi_{s_i}(h_{r^*}) < v_1 + \epsilon_1 \), hence \( \varphi_s(\varphi_{s_i}(h_{r^*})) \to v_1 \) as \( s \to \infty \). Then, for some \( j \) with \( j > s_i \), we have \( \varphi_s(\phi_j(h_{r^*})) = \varphi_{s+j-s_i}(\varphi_{s_i}(h_{r^*})) \to v_1 \) as \( s \to \infty \), so by (4.38) \( r^* < \bar{r}_j \leq r^* \), which is impossible. Thus, \( w \neq v_1 \), so by uniqueness for solutions of ordinary differential equations, we have \( w > v_1 \).

Suppose now that \( w = v_2 \). Then, for some large \( i \), we have \( v_2 - \epsilon_2 < \varphi_{s_i}(h_{r^*}) \leq v_2 \), and so we must have \( \varphi_s(\varphi_{s_i}(h_{r^*})) \to v_2 \) as \( s \to \infty \). But then for some large \( j \) with \( j > s_i \), we must have \( \varphi_s(\phi_j(h_{r^*})) = \varphi_{s+j-s_i}(\varphi_{s_i}(h_{r^*})) \to v_2 \) as \( s \to \infty \). But then by (4.39), \( r^* > \bar{r}_j \geq r^* \) which is impossible. Thus, \( w \neq v_2 \), so \( w < v_2 \).

Finally, we can show that if \( V \) satisfies (DV5)', then we cannot have case (ii) for the \( q_k \) given by Proposition 4.2.10.

**Proposition 4.2.28.** If \( q_k \in E_k \) are the solutions given by Proposition 4.2.10, then \( |\Gamma_k| \) is bounded independently of \( k \).

**Proof.** Suppose not. Then, the proof of Proposition 4.2.17, shows there is a \( k \) such that \( q_k^A < q_k \). But then Lemma 4.2.26 implies that \( \varphi_s(w) \to q_k \) as \( s \to \infty \) for all
Let \( w \in E_k \) with \( q_k^A < w \leq q_k \). Let \( \varepsilon_2 > 0 \) be so small that \( q_k^A < q_k - \varepsilon_2 \). Then, if \( q_k - \varepsilon_2 < w \leq q_k \), we have \( \varphi_s(w) \to q_k \) as \( s \to \infty \). By Lemma 4.2.25, there is an \( \varepsilon_1 \) such that if \( 0 \leq w < \varepsilon_1 \), then \( \varphi_s(w) \to 0 \) as \( s \to \infty \). Thus, by Lemma 4.2.27, there must be a solution \( v \in E_k \) of (HS) such that \( 0 < v < q_k \). But the existence of such a \( v \) contradicts Lemma 4.2.24.

\[ \square \]

\section{4.3 Variational Characterizations}

In this section, we discuss possible variational characterizations of the basic sub-harmonic solutions from Proposition 4.2.10.

\textbf{Definition 4.3.1.} Let

\[ \Gamma_{k,1} := \{ h \in C([0,1], E_k) \mid h(0) \equiv 0, h(1) \equiv 1, 0 \leq h(\theta) \leq 1, \text{ and } h(\theta) \text{ intersects } 1/2 \text{ at most 2 times in } [-k,k] \} \]

and let

\[ c_{k,1} := \inf_{h \in \Gamma_{k,1}} \max_{\theta \in [0,1]} I_k(h(\theta)) \]

Notice that the paths \( h^k_s \) that we use to define the \( q_k \) are elements of \( \Gamma_{k,1} \).

\textbf{Lemma 4.3.2.} \( c_{k,1} \geq \alpha_k \geq a \) where \( \alpha_k \) and \( a \) are from Lemma 4.2.3.

\textit{Proof.} Suppose that \( h \in \Gamma_{k,1} \), and let \( \hat{\theta} \) be the first \( \theta \) at which \( h(\theta) \) intersects \( 1/2 \), and such that \( h(\theta) < 1/2 \) for all \( \theta < \hat{\theta} \). Then, by the definition of \( \alpha_k \) in Lemma 4.2.3, we have \( \alpha_k \leq I_k(h(\hat{\theta})) \leq \max_{\theta \in [0,1]} I_k(h(\theta)) \). Since this is true for all \( h \in \Gamma_{k,1} \), we must have \( \alpha_k \leq c_{k,1} \). \[ \square \]
The interesting question is whether or not the minimax values $c_{k,1}$ are critical values of $I_k$. We cannot use the standard deformation argument from the mountain pass theorem, since it is unclear whether $\Gamma_{k,1}$ is invariant under the flow arising from the gradient of $I_k$. However, due to Proposition 4.2.8, $\Gamma_{k,1}$ is invariant under the (semi)-flow $\varphi_s$ arising from the parabolic PDE. In fact, we will give two proofs that the $c_{k,1}$ are critical points of $I_k$, each using the flow $\varphi_s$. The first approach is very similar to the traditional “deformation via gradient of $I_k$” proof. For simplicity, we assume that (DV5)' holds. Notice that in this case, any non-constant periodic solution $u$ of (HS) with $0 < u < 1$ must cross $1/2$. Let

$$A_k := \{ q \in E_k \mid q \text{ crosses } 1/2 \text{ at most twice and } 0 \leq q \leq 1 \} \quad (4.40)$$

Then, if $u \in A_k$ solves (HS) and is non-constant, $u$ intersects $1/2$. Recall that $K_k(c) := \{ q \in E_k \mid I_k'(q) = 0 \}$. We first need the following lemma:

**Lemma 4.3.3.** Suppose $0 < b$ is such that $A_k \cap K_k(b) = \emptyset$ and $I(1/2) \neq b$. Then, given any $\varepsilon > 0$, there is an $\varepsilon' \in (0, \varepsilon)$ such that $\varphi_1(A_k \cap I_k^{b+\varepsilon}) \subset I_k^{b-\varepsilon}$.

**Proof.** We claim that there is an $\varepsilon > 0$ and a $\delta > 0$ such that $\|I_k'(z)\| \geq \delta$ for all $z \in A_k \cap (I_k)^{b+\varepsilon}_{b-\varepsilon}$. If this is false, there is a sequence $\{ z_n \} \subset A_k$ such that $I_k(z_n) \to b$ and $I_k'(z_n) \to 0$. Since $I_k$ satisfies the (PS) condition, there is a $z$ such that $z_n \to z$ in $E_k$. But then $I_k'(z) = 0$ and $I_k(z) = b$. Therefore, $z \neq 1/2$, and $z \neq 0$. Since $z \in \overline{A_k}$ and $z$ solves (HS), $z$ must cross $1/2$ exactly twice. Hence $z \in A_k$, which contradicts the assumption that $A_k \cap K_k(b) = \emptyset$. Without loss of generality, we may assume that $\varepsilon < \varepsilon'$. Let $\varepsilon < \max\{ \delta^2/2, \varepsilon' \}$. By Lemma 4.2.6, if the lemma is false, there must be a $u \in (I_k)^{b+\varepsilon}_{b-\varepsilon}$ such that $I_k(\varphi_1(u)) > b - \varepsilon$. But then $\|I'(\varphi_s(u))\| \geq \delta$.
for all $s \in [0, 1]$. If $g(s) = I_k(\varphi_s(u))$, then $g'(s) \leq -\|I'_k(\varphi_s(u))\|^2 \leq -\delta^2$. (See Lemmas B.7 and B.9 in the appendix). Thus,

$$g(1) = g(0) + \int_0^1 g'(s)\, ds \leq g(0) - \delta^2$$

But since $\varepsilon < \delta^2/2$ and $g(0) \leq b + \varepsilon$, we must have

$$g(1) \leq b + \varepsilon - \delta^2/2 - \delta^2/2 \leq b - \delta^2/2 \leq b - \varepsilon,$$

a contradiction to our assumption about $u$.  

With this lemma in mind, we may prove that $c_{k,1}$ is a critical value of $I_k$ following a standard argument in critical point theory.

**Proposition 4.3.4.** The $c_{k,1}$ defined in Definition 4.3.1 are critical values of $I_k$, and there is a $2k$-periodic solution $q_k$ of (HS) such that $0 < q < 1$, $I_k(q_k) = c_{k,1}$, and $q$ crosses $1/2$ exactly twice.

**Remark:** If we assume only (DV5), we can still show that $c_{k,1}$ is a critical value of $I_k$, but we are not able get as precise a statement about the intersections of the corresponding critical point with $1/2$.

**Proof.** If the proposition is false, then $A_k \cap \mathcal{K}(c_{k,1}) = \emptyset$. Since $0 < c_{k,1} < M$ for all $k$, we must have $I_k(1/2) \neq c_{k,1}$ for all sufficiently large $k$. Let $\hat{\varepsilon} = \frac{c_{k,1}}{2}$, and let $\varepsilon$ be that of Lemma 4.3.3. Pick $h \in \Gamma_{k,1}$ such that $\max_{\theta \in [0,1]} I_k(h(\theta)) < c_{k,1} + \varepsilon$. Now, since $0$ and $1$ solve (HS), $\varphi_s$ fixes the endpoints of $h$. By Proposition 4.2.8, $\varphi_1(h(\theta)) \in \Gamma_{k,1}$. Notice that $h([0,1]) \subset A_k$, and so by Lemma 4.3.3, $\varphi_1(h([0,1])) \subset I_k^{c_{k,1} - \varepsilon}$. But then $\max_{\theta \in [0,1]} \varphi_1(h(\theta)) \leq c_{k,1} - \varepsilon$, a contradiction to the definition of $c_{k,1}$.  

We can also give an alternative proof of Proposition 4.3.4:
Proof. of Prop. 4.3.4 Fix $m \in \mathbb{N}$, and let $h \in \Gamma_{k,1}$ be such that $\max_{\theta \in [0,1]} I_k(h(\theta)) < c_{k,1} + 1/m$. Consider a sequence $s_i \to \infty$ as $i \to \infty$. For each $i$ and $h \in \Gamma_{k,1}$, let $\theta_i$ be such that $I_k(\varphi_{s_i}(h(\theta))) = \max_{\theta \in [0,1]} I_k(\varphi_{s_i}(h(\theta)))$.

Then, on a subsequence, $\theta_i \to \hat{\theta}$ as $i \to \infty$. We know that on a subsequence $\varphi_{s_i}(h(\hat{\theta})) \to \bar{u}$ as $i \to \infty$. We claim that $I_k(\bar{u}) \geq c_{k,1}$. If not, then for some sufficiently large $i$, $I_k(\varphi_{s_i}(h(\hat{\theta}))) < c_{k,1}$. Since $h(\theta_j) \to h(\hat{\theta})$ as $j \to \infty$, there is a large $j > i$ such that $I_k(\varphi_{s_j}(h(\theta_j))) < c_{k,1}$. Letting $s_i$ increase to $s_j$ and noting that $I_k$ decreases along $\varphi_s$, we would have $I_k(\varphi_{s_j}(h(\theta_j))) = \max_{\theta \in [0,1]} I_k(\varphi_{s_j}(h(\theta))) < c_{k,1}$, which is impossible. Next, we show that $\bar{u}$ crosses 1/2 exactly twice. Since $c_{k,1} > 0$, $\bar{u} \neq 0$. Moreover, by Lemma 4.3.5, $\bar{u} \neq 1/2$, so $\bar{u}$ must cross 1/2. Notice that $h(\hat{\theta})$ crosses 1/2 at most twice, since $h \in \Gamma_{k,1}$, hence $\varphi_{s_i}(h(\hat{\theta}))$ crosses 1/2 at most twice. Therefore, since $\bar{u}$ is the uniform limit of curves that intersect 1/2 at most twice, $\bar{u}$ must cross 1/2 at most twice. Therefore, $\bar{u}$ crosses 1/2 exactly twice.

Therefore, we must have $c_{k,1} + 1/m \geq I_k(\bar{u}) \geq c_{k,1}$. If $I_k(\bar{u}) > c_{k,1}$, then we can find a sequence $\bar{u}_m \in E_k$ of solutions of (HS) such that $c_{k,1} < I_k(\bar{u}_m) < c_{k,1} + 1/m$ and each $\bar{u}_m$ crosses 1/2 exactly twice. By the same arguments which we have used before, there is a $\bar{u}$ such that $\bar{u}_m \to \bar{u}$ in $C^2([-k,k])$. Thus, $\bar{u}$ solves (HS), and $c_{k,1} = I_k(\bar{u})$. In addition, since $\bar{u}$ is the uniform limit of curves that intersect 1/2 exactly twice, $\bar{u}$ crosses 1/2 exactly twice. 

Notice that both of these proofs rely to a great extent on the fact that $I_k$ satisfies the Palais-Smale condition. The first proof parallels the “classic” proof of the mountain pass theorem. The second relies on the compactness of the $\omega$-limit set of the flow $\varphi_s$, which (as can be seen in the appendix) in turn hinges on the
Palais-Smale condition.

In some sense, the first proof is somewhat indirect: if $c_{k,1}$ is not a critical value, then we get a contradiction to the definition of $c_{k,1}$, but we do not actually construct a critical point $v$ with $I_k(v) = c_{k,1}$. The second proof is more constructive: we pick a “good” initial condition, and then use the flow to get a critical value. This has the advantage that one can use information about the initial condition to make statements about the corresponding solution of (HS). For the proof of Theorem 4.1.1, we needed to use the special structure of the initial condition $q_k^A$ to conclude the existence of a subsequence $k_j$ for which the amount of time where $q_{k_j}$ is larger than 1/2 was bounded independently of $j$.

We may also generalize the previous discussion. Let

$$
\Gamma_{k,n} := \{ h \in C([0, 1], E_k) \mid h(0) \equiv 0, \ h(1) \equiv 1, \ 0 \leq h(\theta) \leq 1 \text{ and } h(\theta) \text{ crosses } 1/2 \text{ at most } 2n \text{ times in } [-k, k] \}.
$$

Then, we may define

$$
c_{k,n} := \inf_{h \in \Gamma_{k,n}} \max_{\theta \in [0, 1]} I_k(h(\theta)).
$$

Notice that $h_s^k \in \Gamma_{k,n}$ for all $n$. Arguing exactly as above, we can show that the $c_{k,n}$ are critical values of $I_k$. Moreover, we have $c_{k,n} \geq c_{k,n+1}$. Let $q_{k,n}$ be a critical point of $I_k$ corresponding to the values $c_{k,n}$. We have not established whether or not the functions $q_k$ as given by Proposition 4.2.10 equals any of the $q_{k,n}$. At present, we do not know how to answer this question, absent some type of uniqueness.

If we wish only to investigate the existence of homoclinics to 0, we would much rather know whether or not the amount of time any of the $q_{k,n}$ spends larger than 1/2 is bounded or not. It is not clear whether or not there is a subsequence $q_{k_j,n}$.
for which the amount of time that $q_{k,n}$ is larger than 1/2 is bounded. The reason
that we are able to make this statement for the $q_k$ is because of the extra structure
we have for the initial condition $q^A_k$. However, if we make the assumption that the
only zeros of $V_q(t, x)$ are $x = 0, 1/2$ or 1 (i.e. (DV5)'), then we can show that the
amount of time that $q_{k,1}$ spends larger than 1/2 is bounded independently of $k$.
To prove this, let us first give an analogue of Proposition 4.2.10, (1):

**Lemma 4.3.5.** There is an $M$ such that $c_{k,n} \leq M$ for all $k, n$.

Notice that we will only be interested in the $n = 1$ case.

*Proof.* Notice that we must have

$$c_{k,n} \leq \max_{s \in [0, 1]} I_k(h_s^k),$$  \hspace{1cm} (4.41)

since $h_s^k \in \Gamma_{k,n}$ for all $k, n$. But Lemma 4.2.4 implies that

$$\max_{s \in [0, 1]} I_k(h_s^k) \leq C,$$  \hspace{1cm} (4.42)

which finishes the proof. \hfill \Box

By replacing $q_{k,1}(t)$ by $q_{k,1}(t + j)$ for an appropriate $j \in \mathbb{Z}$, we may assume that
there is a maximal interval $(a_k, b_k)$ with midpoint in $[0, 1]$ such that for $t \in (a_k, b_k)$,
$q_{k,1}(t) > 1/2$. Next, for any $\eta < 1/4$, we define the following sets:

$$S_{k,1} := \{ t \in [b_k - 2k, a_k] \mid q_{k,1}(t) \in [0, \eta) \}$$

$$S_{k,2} := \{ t \in [b_k - 2k, a_k] \mid q_{k,1}(t) \in [\eta, 1/2 - \eta] \}$$

$$S_{k,3} := \{ t \in [b_k - 2k, b_k] \mid q_{k,1}(t) \in (1/2 - \eta, 1/2 + \eta) \}$$  \hspace{1cm} (4.43)

$$S_{k,4} := \{ t \in [a_k, b_k] \mid q_{k,1}(t) \in [1/2 + \eta, 1 - \eta] \}$$

$$S_{k,5} := \{ t \in [a_k, b_k] \mid q_{k,1}(t) \in (1 - \eta, 1] \}$$
Thus, we have $|S_{k,1}| + |S_{k,2}| + |S_{k,3}| + |S_{k,4}| + |S_{k,5}| = 2k$. Notice that for any $0 < \eta < 1/4$, Corollary 4.2.23 implies that $|S_{k,2}| + |S_{k,4}|$ is bounded independently of $k$. Let us assume now that

$$\left\{ t \in [b_k - 2k, b_k] \mid q_{k,1}(t) \geq \frac{7}{8} \right\} \neq \emptyset \text{ for all large } k. \tag{4.44}$$

Notice that if this is not the case, then for all large $k$ and $\eta < 1/8$, $S_{k,5} = \emptyset$. Since we must have $|S_{k,2}| + |S_{k,4}|$ bounded independently of $k$, if $|S_{k,3}|$ is unbounded in $k$, then by taking $\eta$ suitably small that $-V(t, q) > -\frac{c_2}{2}$ for all $q \in (1/2 - \eta, 1/2 + \eta)$, we will have a contradiction to Lemma 4.3.5. Therefore, if (4.44) does not hold, then we are in the case where the amount of time that $q_{k,1}$ spends greater than $1/2$ is bounded independent of $k$.

Our goal is to show that $|S_{k,5}|$ is bounded independently of $k$. As a first step, let us show that $|S_{k,3}|$ is bounded independently of $k$. First, we need some technical lemmas about the behavior of $q_{k,1}$ near $a_k, b_k$.

**Lemma 4.3.6.** There is a $c > 0$ independent of $k$ such that $\dot{q}_{k,1}(a_k) \geq c > 0 > -c \geq \dot{q}_{k,1}(b_k)$.

**Proof.** From Lemma 4.2.14, there is a $\Lambda$ such that

$$\left| \left\{ t \in (a_k, b_k) \mid q_{k,1}(t) \in \left( \frac{5}{8}, \frac{7}{8} \right) \right\} \right| < 2\Lambda \tag{4.45}$$

By (4.44), there are $c_k < d_k \in (a_k, b_k)$ be such that $q_{k,1}(c_k) = 5/8$, $q_{k,1}(d_k) = 7/8$, and $q_{k,1}(t) \in (5/8, 7/8)$ for $t \in [c_k, d_k]$. Next, notice that Lemma 4.2.21 applies to the sequence of solutions $q_{k,1}$, since the $q_{k,1}$ are solutions of (HS) that cross $1/2$ at most twice in $[-k, k]$ and we are assuming (DV5)’’. Thus, we will have

$$\frac{1}{4} = q_{k,1}(d_k) - q_{k,1}(c_k) = \int_{c_k}^{d_k} \dot{q}_{k,1}(s) ds \leq \dot{q}_{k,1}(a_k)(d_k - c_k) \leq 2\Lambda \dot{q}_{k,1}(a_k). \tag{4.46}$$
Therefore, we have \( 0 < \frac{1}{4} \leq 2\Lambda \dot{q}_{k,1}(a_k) \), and thus
\[
0 < \frac{1}{8\Lambda} \leq \dot{q}_{k,1}(a_k).
\]
A similar bound is found for \( \dot{q}_{k,1}(b_k) \).

**Lemma 4.3.7.** There is an \( L \) independent of \( k \) such that if \( t \in (a_k - L, a_k + L) \), then \( \dot{q}_{k,1}(t) \geq c/2 \). Similarly, if \( t \in (b_k - L, b_k + L) \), then \( \dot{q}_{k,1}(t) \leq -c/2 \).

**Proof.** Since the functions \( q_{k,1} \) satisfy (HS) and \( 0 \leq q_{k,1} \leq 1 \), we have \( \|\dot{q}_{k,1}\|_{L^\infty(\mathbb{R})} \leq K \) for some \( K \) independent of \( k \). Suppose now that \( |t| \leq \frac{c}{2K} =: L \). Then, we have
\[
\dot{q}_{k,1}(a_k + t) = \dot{q}_{k,1}(a_k) + \int_{a_k}^{a_k+t} \ddot{q}_{k,1}(s) \, ds 
\geq c - \left| \int_{a_k}^{a_k+t} \ddot{q}_{k,1}(s) \, ds \right|
\geq c - |t|K \geq \frac{c}{2},
\]
using Lemma 4.3.6.

A similar calculation provides us the analogous statement for \( b_k \).

Finally, we can show that \( |S_{k,3}| \) is bounded.

**Lemma 4.3.8.** There is an \( \eta_1 > 0 \) such that for all \( \eta < \eta_1 \), \( |S_{k,3}| \) is bounded depending only on \( \eta \).

**Proof.** Notice that \( S_{k,3} \) is the disjoint union of three intervals: \([b_k - 2k, c_k]\) where \( \dot{q}_{k,1}(t) < 0 \) and \( q_{k,1} \) moves from \( 1/2 \) to \( 1/2 - \eta \), \((d_k, e_k)\) where \( \dot{q}_{k,1}(t) > 0 \) and \( q_{k,1}(t) \) moves from \( 1/2 - \eta \) to \( 1/2 + \eta \) and finally \((f_k, b_k]\) where \( \dot{q}_{k,1}(t) < 0 \) and \( q_{k,1} \) decreases from \( 1/2 + \eta \) to \( 1/2 \) (see the diagram after Lemma 4.2.21). If \([b_k - 2k, c_k]\)
is unbounded, then we have for all large enough $k$

$$
\eta = q_{k,1}(b_k - 2k) - q_{k,1}(c_k) = \int_{c_k}^{b_k - 2k} \dot{q}_{k,1}(s)ds \tag{4.47}
$$

$$
\geq - \int_{b_k - 2k}^{b_k - 2k + L} \dot{q}_{k,1}(s)ds \geq \frac{Lc}{2} > 0,
$$

which is impossible if $2\eta < Lc$. An analogous argument shows that $(f_k, b_k)$ is bounded. Similarly, if $(d_k, e_k)$ is unbounded, then

$$
2\eta = q_{k,1}(e_k) - q_{k,1}(d_k) = \int_{d_k}^{e_k} \dot{q}_{k,1}(s)ds \tag{4.48}
$$

$$
\geq \int_{a_k - L}^{a_k + L} \dot{q}_{k,1}(s)ds \geq Lc,
$$

which is impossible if $\eta < \frac{Lc}{2}$. Thus, we may take $\eta_1 := \frac{Lc}{2}$.

Next, we show that the contribution to $I_k(q_{k,1})$ from $S_{k,1}$ is bounded, regardless of the length of $S_{k,1}$, and the contribution to $I_k(q_{k,1})$ from $S_{k,5}$ is comparable to $-\lambda |S_{k,5}|$ for some $\lambda > 0$. Thus, in order for $I_k(q_{k,1})$ to be positive, $|S_{k,1}|$ will have to be bounded. To do this, we use the following 1-dimensional version of the maximum principle:

**Lemma 4.3.9.** Let

$$
L := -\frac{d^2}{dt^2} + a \tag{4.49}
$$

where $a > 0$ and suppose that $g \in C^2(\alpha, \beta)$ satisfies

$$
Lg(t) \geq 0 \quad \text{for } t \in (\alpha, \beta) \text{ and } g(\alpha), g(\beta) \geq 0.
$$

Then, $g(t) \geq 0$ for all $t \in [\alpha, \beta]$. 

Proof. For completeness, we include the proof. If the conclusion is false, then there is a \( t^* \in (\alpha, \beta) \) such that \( g(t^*) < 0 \). Therefore, \( \min_{t \in [\alpha, \beta]} g(t) < 0 \). Let \( \bar{t} \) be a point at which the minimum is achieved. Then, we must have \( g(\bar{t}) < 0 \) and \( \dot{g}(\bar{t}) = 0 \). Since \( \bar{t} \) is a minimizer for \( g \), we must have \( \ddot{g}(\bar{t}) \geq 0 \). Thus, at \( \bar{t} \), we will have

\[
Lg(\bar{t}) = -\ddot{g}(\bar{t}) + ag(\bar{t}) < ag(\bar{t}) < 0
\]

which is impossible. \( \square \)

Now, we let \( a > 0 \) be free for the moment. Let \( q \) be a solution of (HS) with \( q(\alpha) = \eta = q(\beta) \) and \( q(t) \in [0, \eta] \) for all \( t \in (\alpha, \beta) \). Then

\[
\frac{d}{dt} q^2(t) = 2q(t)\dot{q}(t)
\]

and

\[
\frac{d^2}{dt^2} q^2(t) = 2\ddot{q}(t) + 2q(t)\dot{q}(t) = 2\ddot{q}(t) - 2q(t)V_q(t, q(t)).
\]

Thus, (4.52) implies that

\[
L(q^2)(t) = -\frac{d^2}{dt^2} q^2(t) + aq^2(t) = -2q^2(t) + 2q(t)V_q(t, q(t)) + aq^2(t)
\]

But by the assumption that \( 0 \) is a nondegenerate maximum of \( V \) for all \( t \), there is an \( \eta > 0 \) and a \( \beta_1 > 0 \) such that \( xV_q(t, x) \leq -\beta_1 x^2 \) for all \( x \in [-\eta, \eta] \). Thus, (4.53) implies that

\[
L(q^2)(t) \leq -2q^2(t) + (a - 2\beta_1)q^2(t) \leq 0
\]
if $a < \beta / 2$. Now the unique solution of $Lu = 0$ for $t \in (\alpha, \beta)$ and $u(\alpha), u(\beta) = \eta$ is

$$u(t) := \frac{\eta}{\left(e^{\sqrt{a}(\beta-\alpha)} - e^{-\sqrt{a}(\beta-\alpha)}\right)} \left( e^{\sqrt{a}(t-\alpha)} - e^{-\sqrt{a}(t-\beta)} + e^{-\sqrt{a}(t-\beta)} - e^{-\sqrt{a}(t-\alpha)} \right)$$

(4.55)

Notice then that we must have

$$L(u - q^2) = Lu - L(q^2) = 0 - L(q^2) \geq 0$$

(4.56)

Moreover, if $\eta < 1$ we also have $u(\alpha) > q^2(\alpha), u(\beta) > q^2(\beta)$, and so Lemma 4.3.9 implies that

$$q^2(t) \leq u(t) \text{ for all } t \in [\alpha, \beta].$$

(4.57)

Notice that if $\eta > 0$ is suitably small, then there is a $\beta_2 > 0$ such that

$$-V(t, x) \leq \beta_2 x^2 \text{ for all } x \in [-\eta, \eta].$$

(4.58)

Therefore,

$$\int_{\alpha}^{\beta} -V(t, q(t))dt \leq \beta_2 \int_{\alpha}^{\beta} u(t)dt.$$  

(4.59)

But a straightforward calculation shows that

$$\int_{\alpha}^{\beta} u(t)dt = \frac{2\eta}{\sqrt{a}(e^{\sqrt{a}(\beta-\alpha)} - e^{-\sqrt{a}(\beta-\alpha)})} \left( e^{\sqrt{a}(\beta-\alpha)} - 2 + e^{-\sqrt{a}(\beta-\alpha)} \right),$$

(4.60)

and this is bounded as $\beta - \alpha \to \infty$. Thus, if $q$ solves (HS) and $(\alpha, \beta)$ is any maximal interval on which $q(t) \in [0, \eta]$ for all $t \in [\alpha, \beta]$, then there is a $C_1 = C_1(\eta)$ such that

$$\int_{\alpha}^{\beta} -V(t, q(t))dt \leq C_1(\eta).$$

(4.61)
Next, we show that there is $C_2$ depending on the $L^\infty$ bounds of $q$ such that
\[ \int_\alpha^\beta \dot{q}^2(t) \, dt \leq C_2 \] (4.62)
whenever $q$ solves (HS), $0 \leq q(t) \leq \eta$ for all $t \in [\alpha, \beta]$ and $q$ is bounded in $\mathbb{R}$. An integration by parts reveals
\[ \int_\alpha^\beta \dot{q}^2(t) \, dt = q(t)\dot{q}(t)\bigg|_\alpha^\beta - \int_\alpha^\beta \ddot{q}(t)q(t) \, dt \]
\[ = q(t)\dot{q}(t)\bigg|_\alpha^\beta + \int_\alpha^\beta V(q(t), q(t))q(t) \, dt. \] (4.63)

But, since 0 is a non-degenerate maximum of $V$, there is a $\beta_1 > 0$ such that $xV(t, x) \leq -\beta_1 x^2$ for all $x \in [-\eta, \eta]$. So (4.63) implies that
\[ \int_\alpha^\beta \dot{q}^2(t) \, dt \leq q(t)\dot{q}(t)\bigg|_\alpha^\beta - \beta_1 \int_\alpha^\beta q^2(t) \, dt \]
\[ \leq q(\beta)\dot{q}(\beta) - q(\alpha)\dot{q}(\alpha). \] (4.64)

Since $q$ is bounded in $L^\infty$, (HS) implies that $\ddot{q}$ is also bounded in $L^\infty$, and then Lemma 4.2.1 implies that $\dot{q}$ is bounded in $L^\infty$. Therefore, the last term in (4.64) is bounded. Notice that we have now shown
\[ \int_{S_{k,1}} \frac{1}{2}(\dot{q}_{k,1}(t))^2 - V(t, q_{k,1}(t)) \, dt \leq C_1(\eta) + \frac{C_2}{2}. \] (4.65)

since $S_{k,1}$ is the maximal subinterval of $[b_k - 2k, b_k]$ for which $q_{k,1}(t) \in [0, \eta]$.

We next want to apply a similar argument to show that there is a $\lambda > 0$ and a constant $c_3$ such that for all suitably small $\eta > 0$, we have
\[ \int_{S_{k,5}} \frac{1}{2}(\dot{q}_{k,1}(t))^2 - V(t, q_{k,1}(t)) \, dt \leq -\lambda|S_{k,5}| + c_3. \] (4.66)
Assuming (4.66), then Lemma 4.3.8 and (4.65) imply that \(|S_{k,5}|\) is bounded independently of \(k\), for otherwise (4.66) implies that \(I_k(q_{k,1}) \to -\infty\) as \(k \to \infty\), contradicting Lemma 4.3.2.

To prove (4.66), suppose \(q\) is a solution of (HS) such that \(q(t) \in [1 - \eta, 1]\) for all \(t \in (\alpha, \beta)\). Integrating by parts, we see that
\[
\int_{\alpha}^{\beta} \dot{q}^2(t) \, dt = \dot{q}(t)(q(t) - 1) \bigg|_{\alpha}^{\beta} - \int_{\alpha}^{\beta} \ddot{q}(t)(q(t) - 1) \, dt \quad (4.67)
\]
\[
= \dot{q}(t)(q(t) - 1) \bigg|_{\alpha}^{\beta} + \int_{\alpha}^{\beta} V_q(t, q(t))(q(t) - 1) \, dt.
\]
But since 1 is a non-degenerate maximum for \(V\), there is a \(\beta' > 0\) such that
\[
(x - 1)V_q(t, x) \leq -\beta'(x - 1)^2, \quad (4.68)
\]
for all \(x \in [1 - \eta, 1 + \eta]\), so the integral in the last term of (4.67) is negative. Thus,
\[
\int_{\alpha}^{\beta} q^2(t) \, dt \leq \dot{q}(t)(q(t) - 1) \bigg|_{\alpha}^{\beta} \quad (4.69)
\]
Since \(|q - 1| \leq \eta\), estimating as in (4.64) and using (HS) and Lemma 4.2.1 shows that \(\int_{\alpha}^{\beta} q^2(t) \, dt\) is bounded independently of \(k\).

Next, we have
\[
\frac{d}{dt}(q(t) - 1)^2 = 2(q(t) - 1)\dot{q}(t) \quad (4.70)
\]
and
\[
\frac{d^2}{dt^2}(q(t) - 1)^2 = 2q^2(t) + 2(q(t) - 1)\ddot{q}(t) \quad (4.71)
\]
\[
= 2q^2(t) - 2(q(t) - 1)V_q(t, q(t)).
\]
But then we have
\[
L(q - 1)^2(t) = -2q^2(t) + 2(q(t) - 1)V_q(t, q(t)) + a(q(t) - 1)^2. \quad (4.72)
\]
Combining (4.68) and (4.72), we have
\[ L(q - 1)^2(t) \leq -2q^2(t) + (a - 2\beta_1')(q(t) - 1)^2 \leq 0 \] (4.73)
if \( a < 2\beta_1' \). Therefore, we have
\[ L(u - (q - 1)^2) = Lu - L((q - 1)^2) \geq 0 \] (4.74)
in \((\alpha, \beta)\). Since \( u = \eta \) at the endpoints of \((\alpha, \beta)\), we have \( u - (q - 1)^2 \geq 0 \) at the endpoints. Thus, Lemma 4.3.9 implies that
\[ (q(t) - 1)^2 \leq u(t) \] for all \( t \in (\alpha, \beta) \). (4.75)
Now, there is a \( \beta_2' > 0 \) such that
\[ V(t, x) \geq -\beta_2'(x - 1)^2 + c_1, \] (4.76)
for all \( x \in [1 - \eta, 1 + \eta] \) and so we have
\[ -V(t, q(t)) \leq \beta_2'(q(t) - 1)^2 - c_1 \leq \beta_2' u(t) - c_1. \] (4.77)
But then we must have
\[ \int_\alpha^\beta -V_q(, q(t)) \leq \beta_2' \int_\alpha^\beta u(t)dt - c_1(\beta - \alpha) \] (4.78)
\[ \leq C'(\eta) - c_1(\beta - \alpha), \]
which gives us (4.66).

Thus, if \( (DV5)' \) holds, then for some suitably small fixed \( \eta > 0 \), there is a \( C_4 = C_4(\eta) \) such that
\[ \sum_{i=2}^5 |S_{k,i}| \leq C_4(\eta) \] (4.79)
for all \( k \). Therefore, we are in case (i) of Remark 4.2.20, and so there is a subsequence \( q_{k,j,1} \) that converges in \( C_2^2(\mathbb{R}) \) to a solution \( q_1 \) of (HS) homoclinic to 0.
4.4 Multi-bump type solutions of (HS)

In this section, we investigate the existence of solutions to (HS) that have multiple bumps, in contrast to the single bump of the homoclinic whose existence is given by Theorem 4.1.1. As pointed out in the Introduction, what we will find are not quite multi-bumps, but something richer. Notice that (DV1-6) (or (DV5)’) allow the possibility that the potential $V$ is independent of $t$, and for this autonomous case, there are no multi-bump solutions, as a phase plane argument shows.

Notice that the picture above implies that as the period $k$ increases, the solutions $q_k$ assume values that get closer and closer to 1. More precisely, if $\hat{t}_k$ is a point at which $q_k(t)$ is closest to 1, then $q_k(\hat{t}_k)$ is strictly increasing in $k$. Pictorially, we have
Moreover, as the period increases, the functions $q_k$ assume values that are closer to 0. Therefore, if $k_1 < k_2$, there are points such that $\tau_{r_1} q_{k_1}(t) = \tau_{r_2} q_{k_2}(t)$ for any $r_1, r_2 \in \mathbb{R}$. Since there are no multi-bumps in the autonomous case, one direction suggested by this discussion is to exploit some ordering properties of solutions of (HS).

**Definition 4.4.1.** We say that $V$ and the associated equation (HS) satisfies Condition (MB)

if there is a $k^* \in \mathbb{N}$ and a nonconstant periodic solution $w \in E_{k^*}$ with $0 < w < 1$ and an interval $(a, b)$ such that $w > q_{l_{k^*}}$ and $\Gamma_{l_{k^*}} \subset (a, b)$ for all $l \geq 2, l \in \mathbb{N}$.

**Remark:** (1) Since $w$ is nonconstant, there must be $t_1$ such that $w(t_1) \in (0, 1/2)$ and a $t_2$ such that $w(t_2) > 1/2$.

(2) Without loss of generality, we may assume that $(a, b)$ is maximal with respect to $w(t) > 1/2$ for $t \in (a, b)$. Notice that we must have $b - a < 2k$, for otherwise $w \geq 1/2$, which is impossible if $w$ is nonconstant (because of (DV5)).

Notice that the autonomous case is excluded, since any periodic solution of (HS) must cross any other periodic solution. From the proof of Proposition 4.2.17
in the last section, Condition (MB) will be satisfied if for example there is a sequence $k_j$ such that $\Gamma_{k_j} \to \infty$, as the proof of Lemma 4.2.17 shows. However, if (DV5)’ is satisfied, then it is not at all clear that Condition (MB) is ever satisfied. However, we shall give an example in the following section showing that (MB) may be satisfied, even if (DV5)’ is satisfied.

Our goal is to prove the following:

**Theorem 4.4.2.** If $V$ satisfies (MB), then for every $m \in \mathbb{N}$ and every $Z = \{n_1, n_2, \ldots, n_m\} \in \mathbb{Z}^m$ where $n_1 < n_2 < \cdots < n_m$, there is a solution $\gamma_Z$ of (HS) such that

- $\gamma_Z(t), \dot{\gamma}_Z(t) \to 0$ as $t \to \pm\infty$.

- $\gamma_Z$ intersects 1/2 exactly $2m$ times, twice in each interval $(a + 2n_i k, b + 2n_i k)$, where $a, b$ are from Condition (MB).

Notice that since $\gamma_Z$ intersects 1/2 exactly twice in $(a + 2n_i k, b + 2n_i k)$ for $i = 1, 2, \ldots, m$ and nowhere else, the idea is that $\gamma_Z$ has $m$ “bumps” where $\gamma_Z$ is larger than 1/2, and so looks like $m$ copies of the “basic” homoclinic $q$ from the preceding section. We build $\gamma_Z$ as in the preceding section by constructing a sequence of subharmonic solutions to (HS) that have the same sort of intersections with 1/2 as we would like for $\gamma_Z$. The subharmonic solutions are constructed by choosing an initial condition with the correct intersection, and then using the flow $\varphi_s$ to get a solution of (HS). In order to guarantee that these subharmonics do not gain intersection behavior with 1/2, we use Proposition 4.2.8. However, it could happen that the number of intersections with 1/2 decreases. This is where Condition (MB) plays a role: because of the choice of initial condition, we are
able to use information about $w$ to make sure that intersections with $1/2$ do not disappear. We now suppose that $Z = \{n_1, n_2, \ldots, n_m\}$ is fixed, and Condition (MB) is satisfied.

For every $l$, we let

$$\alpha_l(t) := \max\{\tau_{2n_1k}q_{lk}(t), \tau_{2n_2k}q_{lk}(t), \ldots, \tau_{2n_mk}q_{lk}(t)\} \in E_{lk}.$$ 

The following lemma is immediate from the definition of $\alpha_l$ and the fact that we have Condition (MB):

**Lemma 4.4.3.** We have

- $\alpha_l < w$
- $\alpha_l \geq \tau_{2n_i,k}q_{lk}$ for $i = 1, 2, \ldots, m$

We will use the $\alpha_l$ to build a sequence of subharmonics, each of which has the same sort of intersections with $1/2$ as we want for $\gamma_Z$.

Recall that $\Gamma_n = (a_n, b_n)$ is a maximal interval on which $q_n(t) > 1/2$, and the midpoint of $\Gamma_n$ is in $[0, 1)$.

**Lemma 4.4.4.** We have

(i) $[a_{lk} + 2n_jk, b_{lk} + 2n_jk] \subset (a + 2n_jk, b + 2n_jk)$

(ii) $(a_{lk} + 2n_i k, b_{lk} + 2n_i k) \cap (a_{lk} + 2n_j k, b_{lk} + 2n_j k) = \emptyset$ for $i \neq j$.

(iii) There is an $l_0 = l_0(Z)$ such that for $l \geq l_0$, we have $-lk < a + 2n_1k$ and $b + 2n_mk < lk$. 
Proof. To prove (i), notice that \([a_{lk}, b_{lk}] \subset (a, b)\) because Condition (MB) is satisfied, and so (i) follows.

Notice that (i) implies (ii) if we can show that \((a + 2n_jk, b + 2n_jk) \cap (a + 2n_i k, b + 2n_i k) = \emptyset\) for \(i \neq j\). But if this intersection is nonempty, the sets must be the same (since they are maximal intervals on which \(w(t) > 1/2\)), which is impossible.

Finally, for (iii), let \(l_0 := \max\{2 - 2n_1, 3 + 2n_m\}\). Then we have \(-lk \leq 2n_1 k - 2k\). But \(a > -2k\) since \(b - a < 2k\) and \((a, b)\) contains points in \([0, 1)\). Thus, \(-lk \leq 2n_1 - 2k < a + 2n_1 k\). Similarly, we have \(3k + 2n_m k \leq lk\). Now, \(b < 2k + 1 < 3k\). Thus \(b + 2n_m k < 3k + 2n_m k \leq lk\).

Notice that the preceding lemma and Lemma 4.4.4 imply that if \(l \geq l_0\), then we have \((a_{lk} + 2n_1 k, b_{lk} + 2n_1 k), (a_{lk} + 2n_2 k, b_{lk} + 2n_2 k), \ldots, (a_{lk} + 2n_m, b_{lk} + 2n_m k) \subset [-lk, lk]\). In particular, we have for \(l > l_0\)

\[
\{t \in [-lk, lk] \mid \tau_{2n_i k q_{lk}}(t) > \frac{1}{2}\} = (a_{lk} + 2n_i k, b_{lk} + 2n_i k)
\]

The next lemmas tell us about the sets where \(\alpha_l(t) > 1/2\) and the intersection properties of \(\alpha_l\) with \(1/2\).

**Lemma 4.4.5.** For all \(l > l_0\), we have \(\alpha_i(t) = \tau_{2n_i k q_{lk}}(t)\) if \(t \in [a_{lk} + 2n_i k, b_{lk} + 2n_i k]\) for \(i = 1, 2, \ldots, m\).

*Proof.* Suppose that \(t \in [a_{lk} + 2n_i k, b_{lk} + 2n_i k]\). Then we have \(\tau_{2n_i k q_{lk}}(t) \geq 1/2\). By Lemma 4.4.4, since \(l > l_0\), we have

\[
\{t \in [-lk, lk] \mid \tau_{2n_j k q_{lk}}(t) \geq \frac{1}{2}\} = [a_{lk} + 2n_j k, b_{lk} + 2n_j k]
\]

and \(\tau_{2n_j k q_{lk}}(t) < 1/2 \leq \tau_{2n_i k q_{lk}}(t)\) for \(j \neq i\), and so \(\alpha_i(t) = \tau_{2n_i k q_{lk}}(t)\). \(\square\)
Lemma 4.4.6. For any 
\[ t \in [-lk, lk] \setminus (\bigcup_{i=1}^{m} [a_{lk} + 2ni, b_{lk} + 2ni]), \]
\[ \alpha_i(t) < 1/2. \]

Proof. By Lemma 4.4.5, for \( t \in [-lk, lk] \) and \( t \notin \bigcup_{i=1}^{m} [a_{lk} + 2ni, b_{lk} + 2ni] \), \( \tau_{2ni}q_{lk}(t) < 1/2 \) for \( i = 1, 2, \ldots, m \). Thus, \( \alpha_i(t) < 1/2 \). \( \square \)

The next lemma tells us that each \( \alpha_i \) has the same sort of intersections with \( 1/2 \) as we want for \( \gamma_Z \).

Lemma 4.4.7. For \( l \geq l_0 \), \( \alpha_i \) intersects \( 1/2 \) exactly twice in \( (a + 2ni, b + 2ni) \), and \( \alpha_i \) intersects \( 1/2 \) exactly \( 2m \) times in \( [-lk, lk] \).

Proof. By Lemma 4.4.5, we have \( \alpha_i(a_{lk} + 2ni) = \tau_{2ni}q_{lk}(a_{lk} + 2ni) = 1/2 \) and \( \alpha_i(b_{lk} + 2ni) = \tau_{2ni}q_{lk}(b_{lk} + 2ni) = 1/2 \). Moreover, for \( t \in (a_{lk} + 2ni, b_{lk} + 2ni) \), we have \( \alpha_i(t) = \tau_{2ni}q_{lk}(t) > 1/2 \). Now, since the intervals \( (a_{lk} + 2ni, b_{lk} + 2ni) \) are disjoint and the only places where \( \alpha_i(t) > 1/2 \) are inside these intervals, the only places where \( \alpha_i(t) = 1/2 \) are when \( t = a_{lk} + 2ni \) or \( t = b_{lk} + 2ni \) for some \( i = 1, 2, \ldots, m \). Since \( l \geq l_0 \), these are all inside \( [-lk, lk] \) by Lemma 4.4.4. \( \square \)

Lemma 4.4.8. For \( l \geq l_0 \), we have \( I_{lk}(\alpha_i) < mI_{lk}(q_{lk}) \leq mC \), where \( C \) is from Lemma 4.2.4.

Proof. We have
\[ I_{lk}(\alpha_i) = \int_{-lk}^{lk} \left( \frac{1}{2}(\dot{\alpha}_i)^2 - V(t, \alpha_i) \right) dt \]
\[ \leq \sum_{i=1}^{m} \int_{-lk}^{lk} \left( \frac{1}{2}(\tau_{2ni}q_{lk})^2 \right) dt + \int_{-lk}^{lk} -V(t, \alpha_i(t)) dt. \]

(4.80) (4.81)
But notice that for \( l \geq l_0 \), Lemma 4.4.4 and Lemma 4.4.5 imply that

\[
\int_{-l_k}^{l_k} -V(t, \alpha(t)) \, dt = \sum_{i=1}^{m} \int_{[a_{ik}+2n_i k, b_{ik}+2n_i k]} -V(t, \tau_{2n_i k q_{lk}(t)}) \, dt \\
+ \int_{[-l_k, l_k] \setminus \bigcup_i [a_{ik}+2n_i k, b_{ik}+2n_i k]} -V(t, \alpha(t)) \, dt \tag{4.82}
\]

Next, notice that

\[
[-l_k, l_k] \setminus \bigcup_i [a_{ik}+2n_i k, b_{ik}+2n_i k] = \\
\bigcup_i \{ t \in [-l_k, l_k] \mid \alpha_i(t) = \tau_{2n_i k q_{lk}(t)} \} \setminus [a_{ik}+2n_i k, b_{ik}+2n_i k].
\]

Thus, we have

\[
\int_{[-l_k, l_k] \setminus \bigcup_i [a_{ik}+2n_i k, b_{ik}+2n_i k]} -V(t, \alpha(t)) \, dt = \int_{\bigcup_i \{ t \in [-l_k, l_k] \mid \alpha_i(t) = \tau_{2n_i k q_{lk}(t)} \} \setminus [a_{ik}+2n_i k, b_{ik}+2n_i k]} -V(t, \alpha(t)) \, dt. \tag{4.83}
\]

Now, the sets \( \{ t \in [-l_k, l_k] \mid \alpha_i(t) = \tau_{2n_i k q_{lk}(t)} \} \setminus [a_{ik}+2n_i k, b_{ik}+2n_i k] \) are not disjoint. Moreover, we must have \(-V(t, \alpha(t)) \geq 0\) for \( t \in \{ t \in [-l_k, l_k] \mid \alpha_i(t) = \tau_{2n_i k q_{lk}(t)} \} \setminus [a_{ik}+2n_i k, b_{ik}+2n_i k], i = 1, 2, \ldots, m \) since by Lemma 4.4.6, \( \alpha_i(t) < 1/2 \) on these sets. Therefore, we have

\[
\int_{\bigcup_i \{ t \in [-l_k, l_k] \mid \alpha_i(t) = \tau_{2n_i k q_{lk}(t)} \} \setminus [a_{ik}+2n_i k, b_{ik}+2n_i k]} -V(t, \alpha(t)) \, dt \\
\leq \sum_{i=1}^{m} \int_{\{ t \in [-l_k, l_k] \mid \alpha_i(t) = \tau_{2n_i k q_{lk}(t)} \} \setminus [a_{ik}+2n_i k, b_{ik}+2n_i k]} -V(t, \alpha(t)) \, dt \tag{4.84} \\
= \sum_{i=1}^{m} \int_{\{ t \in [-l_k, l_k] \mid \alpha_i(t) = \tau_{2n_i k q_{lk}(t)} \} \setminus [a_{ik}+2n_i k, b_{ik}+2n_i k]} -V(t, \tau_{2n_i k q_{lk}(t)}) \, dt \\
\leq \sum_{i=1}^{m} \int_{[-l_k, l_k] \setminus [a_{ik}+2n_i k, b_{ik}+2n_i k]} -V(t, \tau_{2n_i k q_{lk}(t)}) \, dt
\]
since on \([-lk, lk]\ \setminus \{a_{lk} + 2n_i k, b_{lk} + 2n_i k\}, \) \(0 < \tau_{2n_i k q_{lk}} < 1/2,\) and so we must have
\[-V(t, \tau_{2n_i k q_{lk}}(t)) > 0.\]
Combining (4.83) and (4.84), we have
\[
\int_{[-lk, lk] \setminus \cup_i ([a_{lk} + 2n_i k, b_{lk} + 2n_i k])} -V(t, \alpha_i(t)) \, dt
\leq \sum_{i=1}^{m} \int_{[-lk, lk] \setminus [a_{lk} + 2n_i k, b_{lk} + 2n_i k]} -V(t, \tau_{2n_i k q_{lk}}(t)) \, dt
\] (4.85)

Now, if we combine (4.82) and (4.85), we see that
\[
\int_{-lk}^{lk} -V(t, \alpha_i(t)) \, dt \leq \sum_{i=1}^{m} \int_{[a_{lk} + 2n_i k, b_{lk} + 2n_i k]} -V(t, \tau_{2n_i k q_{lk}}(t)) \, dt
\]
\[
+ \sum_{i=1}^{m} \int_{[-lk, lk] \setminus [a_{lk} + 2n_i k, b_{lk} + 2n_i k]} -V(t, \tau_{2n_i k q_{lk}}(t)) \, dt
\] (4.86)

Finally, combining (4.81) and (4.86), we see that
\[
I_{lk}(\alpha_i) \leq \sum_{i=1}^{m} \int_{-lk}^{lk} \left( \frac{1}{2} (\tau_{2n_i k q_{lk}})^2 - V(t, \tau_{2n_i k q_{lk}}) \right) \, dt
\] (4.87)
\[
= \sum_{i=1}^{m} I_{lk}(\tau_{2n_i q_{lk}}) = m I_{lk}(q_{lk}) \leq m C
\] (4.88)
by Proposition 4.2.10.

Now, pick a sequence \(s_i \to \infty.\) By Theorem 4.2.5, there is a subsequence \(s_{i_j}\)
such that \(v_i := \lim_{s_{i_j} \to \infty} \varphi_{s_{i_j}}(\alpha_i) \in E_{lk}\) is a solution of (HS). By Lemmas 4.2.6 and 4.4.8, we must have \(I_{lk}(v_i) < mC\) for all \(l,\) and by Proposition 4.2.8, for \(l > l_0,\) we know that \(v_i\) crosses 1/2 at most \(2m\) times in \([-lk, lk].\) Finally, by uniqueness of solutions to ODEs, we must have
\[
v_i < w\] (4.89)
\[
v_i \geq \tau_{2n_i k q_{lk}} \text{ for } i = 1, 2, \ldots, m.\] (4.90)
Lemma 4.4.9. \( v_l \) intersects \( 1/2 \) exactly twice in each interval \((a + 2n_i k, b + 2n_i k)\), and nowhere else.

Proof. It suffices to show that \( v_l \) intersects \( 1/2 \) at least twice in each interval \((a + 2n_i k, b + 2n_i k)\). Notice that by Lemma 4.4.4, we have \( a + 2n_i k < a_{lk} + 2n_i k < b_{lk} + 2n_i k < b + 2n_i k \). Next, from (4.89) and (4.90),

\[
 v_l(a + 2n_i k) < w(a + 2n_i k) = w(a_k) = 1/2 \quad (4.91)
\]

\[
 1/2 = \tau_{2n_i k} q_{lk}(a_{lk} + 2n_i k) \leq v_l(a_{lk} + 2n_i k). \quad (4.92)
\]

Thus, there is a \( t_{i_1}^l \in (a + 2n_i k, a_{lk} + 2n_i k) \) such that \( v_l(t_{i_1}^l) = 1/2 \). Similarly, we have

\[
 v_l(b + 2n_i k) < w(b + 2n_i k) = w(b) = 1/2 \quad (4.94)
\]

Thus, there is a \( t_{i_2}^l \in [b_{lk} + 2n_i k, b_k + p + 2n_i k] \) such that \( v_l(t_{i_2}^l) = 1/2 \). In other words, there are at least two points in each \((a + 2n_i k, b + 2n_i k)\) such that \( v_l(t) = 1/2 \).

Notice that the preceding lemma implies that \( v_l \) intersects \( 1/2 \) exactly \( 2m \) times: twice in every interval \((a + 2n_i k, b + 2n_i k)\) for \( i = 1, 2, \ldots, m \). The next lemma allows us to apply Lemma 4.2.18 to the sequence \( v_l \).

Lemma 4.4.10. For \( l \geq l_0 \), we have

\[
\{ t \in [-lk, lk] \mid v_l(t) > 1/2 \} \subset [-l_0 k, l_0 k]
\]
Proof. From Lemma 4.4.9, $v_l$ intersects $1/2$ only in $(a + 2n_i k, b + 2n_i k)$. Moreover, since $v_l$ intersects $1/2$ $2m$ times in $[-lk, lk]$, we must have $v_l > \tau_{2n_i k} g_{lk}$ by the maximum principle. Thus, $v_l(t) > 1/2$ for $t \in [a_{lk} + 2n_i k, b_{lk} + 2n_i k] \subset (a + 2n_i k, b + 2n_i k)$. But then, since $l \geq l_0$ implies that $-lk < a + 2n_i k < b + 2n_i k < lk$, we must have $\cup_i (a + 2n_i k, b + 2n_i k) \subset [-l_0 k, l_0 k]$.

We claim that for $l \geq l_0$, $\{t \in [-lk, lk] \mid v_l(t) > 1/2\} \subset \cup_i (a + 2n_i k, b + 2n_i k)$. Suppose not. Then $v_l$ must cross $1/2$ outside $\cup_i (a + 2n_i k, b + 2n_i k)$. But this contradicts Lemma 4.4.9.

Now, we can prove

**Theorem 4.4.11.** If $V$ satisfies (MB), then for any $m \in \mathbb{N}$ and any subset $Z = \{n_1, n_2, \ldots, n_m\} \in \mathbb{Z}^m$ with $n_1 < n_2 < \cdots < n_m$, there is a solution $\gamma_Z$ of (HS) such that

- $\gamma_Z(t), \dot{\gamma}_Z(t) \to 0$ as $t \to \pm \infty$.
- $\gamma_Z$ intersects $1/2$ exactly $2m$ times, twice in each interval $(a + 2n_i k, b + 2n_i k)$, where $a, b$ are from Condition (MB).
- $\gamma_Z < w$

**Proof.** We consider the sequence $v_l$ of periodic solutions to (HS). Note that (i) of Lemma 4.2.18 is satisfied, Lemma 4.4.8 implies that (ii) of Lemma 4.2.18 is satisfied, and finally Lemma 4.4.10 tells us that (iii) of Lemma 4.2.18 holds. Thus, a subsequence of $v_l$ converges to some $\gamma_Z$ in $C^2_{loc}(\mathbb{R})$, and $\gamma_Z(t), \dot{\gamma}_Z(t) \to 0$ as $|t| \to \infty$. Since each $v_l$ is a solution of (HS), $\gamma_Z$ also solves (HS). Moreover, by
we must have $\gamma_Z \leq w$. Since $\gamma_Z(t) \to 0$ as $|t| \to 0$, we have $\gamma_Z < w$. It remains only to show that $\gamma_Z$ has the correct intersection properties.

Notice that for any $t \not\in [-l_0k, l_0k]$, we must have $\gamma_Z(t) \leq 1/2$ since this is true for $v_l$ for all $l$ with $lk > |t|$. Moreover, we must in fact have $\gamma_Z(t) < 1/2$, since if we have equality, then $\gamma_Z$ would be tangent to $1/2$, and so identical to $1/2$. But this is contrary to the asymptotics of $\gamma_Z$. Now, in each $(a + 2n_lk, b + 2n_lk)$, each $v_l$ intersects $1/2$ twice. Since $\gamma_Z$ is the uniform limit of the $v_l$ on this interval, there must be a $t \in (a + 2n_lk, b + 2n_lk)$ where $\gamma_Z(t) \geq 1/2$. In fact, there must be a $t$ at which $\gamma_Z(t) > 1/2$ as above. Thus, $\gamma_Z$ crosses $1/2$ at least twice in $(a + 2n_lk, b + 2n_lk)$. If $\gamma_Z$ crosses $1/2$ more than that, then (since $\gamma_Z$ is the uniform limit of the $v_l$ on bounded intervals) for all large $l$, the same must be true of $v_l$, which contradicts Lemma 4.4.10.

Finally, we end this section by proving the existence of “infinite” bump solutions. More precisely, we have

**Theorem 4.4.12.** Let $\{n_m\} \subset \mathbb{Z}$ be a bi-infinite sequence of integers such that $n_m < n_{m+1}$ for all $m \in \mathbb{Z}$. If Condition (MB) is satisfied, then there is a solution $u$ of (HS) such that $u$ crosses $1/2$ exactly twice in $[a + 2n_mk, b + 2n_mk]$ for all $m \in \mathbb{Z}$, and nowhere else. Moreover, if $\{n_m\} \neq \mathbb{Z}$, we have $u < w$.

**Remark:** If $\{n_m\} = \mathbb{Z}$, then $w$ is an “infinite” bump solution with the intersection properties above, and we might have $u \equiv w$.

**Proof.** For every $l \in \mathbb{N}$, consider

$$\{n_{-l}, n_{-l+1}, \ldots, n_o, \ldots, n_{l-1}, n_l\} \subset \mathbb{Z}^{2l+1}$$
By Theorem 4.4.2, there is a solution $v_l$ of (HS) such that $v_l$ crosses 1/2 exactly twice in $(a + 2n_m k, b + 2n_m k)$ for $m = -l, -l + 1, \ldots, l - 1, l$, and nowhere else.

Notice that each $v_l$ satisfies $0 \leq v_l \leq w$, where $w$ is from Condition (MB). Notice that the sequence $v_l$ is bounded in $C^2$, since each $v_l$ is bounded in $L^\infty(\mathbb{R})$, and thus using (HS) and Lemma 4.2.1, we see that $v_l$ is bounded in $C^2(\mathbb{R})$. Moreover, using (HS), we see that $\{\ddot{v}_l\} \subset C(\mathbb{R})$ is equicontinuous. Thus, there is a subsequence $l_j$ such that $v_{l_j}$ converges to some $u$ in $C^2_{loc}(\mathbb{R})$. By relabeling, we may as well assume that $v_l \to u$. Therefore, $u$ satisfies (HS). We also know that each $v_l < w$, and so $u \leq w$. Next, we show that $u$ has the correct behavior with respect to 1/2.

Fix an $m$. We need to show that $u$ crosses 1/2 exactly twice in $[a + 2n_m k, b + 2n_m k]$. For all sufficiently large $l$, we know that $v_l$ converges uniformly to $u$ in $C^2([a + 2n_m k, b + 2n_m k])$ and each $v_l$ crosses 1/2 exactly twice in this interval. Let $t_l$ be the point in $(a + 2n_m k, b + 2n_m k)$ where $v_l$ is closest to 1. We have

$$|u(\hat{t}) - v_l(t_l)| \leq |u(\hat{t}) - v_l(\hat{t})| + |v_l(\hat{t}) - v_l(t_l)|$$
$$\leq |u(\hat{t}) - v_l(\hat{t})| + \|\dot{v}_l\|_{L^\infty(\mathbb{R})} |\hat{t} - t_l|.$$  

The first term goes to 0 since $v_l$ converges locally uniformly to $u$, and the second goes to zero since $\dot{v}_l$ is uniformly bounded in $L^\infty(\mathbb{R})$ and $t_l \to \hat{t}$ as $l \to \infty$. Thus, we must have $u(\hat{t}) \geq 1/2$. In addition, we have $\dot{v}_l(t_l) = 0$ for all $l$, and so repeating the argument above, we have $\dot{u}(\hat{t}) = 0$. Since $u \leq w$, and there is a $t_1$ for which $w(t_1) < 1/2$, $u \not\equiv 1/2$. Thus, we must have $u(\hat{t}) > 1/2$. Because each of the $v_l$ is smaller than 1/2 outside of the intervals $(a + 2n_m k, b + 2n_m k)$ and the $v_l$ converge to $u$ locally uniformly, we must have $u(t) \leq 1/2$ for $t \not\in \cup_m [a + 2n_m k, b + 2n_m k]$. Therefore, $u$ crosses 1/2 at least twice in $[a + 2n_m k, b + 2n_m k]$, and outside of
these intervals, \( u(t) < 1/2 \). We have only to show that \( u \) cannot cross 1/2 more than twice in \([a + 2n_m k, b + 2n_m k]\). The argument is exactly that in the proof of Theorem 4.4.2, and we do not repeat it here.

\[\square\]

### 4.5 An Example

When is Condition (MB) satisfied? In this section prove that it suffices that (HS) possesses a solution homoclinic to 1.

**Proposition 4.5.1.** Suppose that \( V \) satisfies (DV1-6), and there is a solution \( h \) of (HS) homoclinic to 1 such that \( 0 < h < 1 \). Then, Condition (MB) is satisfied.

**Proof.** Throughout, we assume that \( q_k \) are given by Proposition 4.2.10, and \( q_{k_j} \) is a sequence that converges locally uniformly to \( q \) and for which \( |\Gamma_{k_j}| \) is bounded. Let

\[
\delta(h) := \inf_{t \in \mathbb{R}} h(t) > 0,
\]

so \( \delta(h) \) is the closest that \( h \) comes to 0. Because of the asymptotics of \( h \), there must be a \( \hat{t} \) for which \( h(\hat{t}) = \delta(h) \), and \( \dot{h}(\hat{t}) = 0 \). Because \( h \not\equiv 1/2 \), we must have \( \delta(h) \not= 1/2 \). Suppose that \( \delta(h) > 1/2 \). But then \( \ddot{h}(t) = -V_q(t, h(t)) \) is either always positive or always negative. In fact, since \( q = 1 \) is always a local maxima for \( V(t, q) \), then \( -V_q(t, q) < 0 \), and thus \( \dot{h}(t) < 0 \) for all \( t \in \mathbb{R} \). Therefore, \( \dot{h}(t) \) is decreasing, and for \( t > \hat{t}, \dot{h}(t) < 0 \). But then for \( t > \hat{t}, h(t) < h(\hat{t}) \), which is impossible. Thus, \( \delta(h) < 1/2 \).

Next, note that there must be a \( \delta(q) \in (0, 1/2) \) such that \( q_{k_j}(t) < 1 - \delta(q) \) for all \( k_j \). Indeed, if there were no such \( \delta(q) > 0 \), then the \( q_{k_j} \) would get arbitrarily close
to 1, and so have to spend longer and longer intervals close to 1, which contradicts
the fact that $|\Gamma_k|$ is bounded. Notice that we must then have $q \leq 1 - \delta(q)$.

Recall that by Proposition 4.2.10, for all $k \geq k_0$, \{ $t \in [-k,k] \mid q_k^A(t) \neq 0$ \}
is an interval centered at the origin whose length is bounded independently of $k$.
Thus, there is $M$ such that \{ $t \in [-k,k] \mid q_k^A(t) \neq 0$ \} $\subset [-M,M]$ for all $k \geq k_0$.
Furthermore, we know that $q_k^A(t) \leq \eta$ for all $k \geq k_0$.

Let $c$ be such that $c > \max\{1 - \delta(q)/2, \eta\}$. Now, pick $a_h \in Z$ such that
for $t > -M - 1$, $\tau_{a_h} h(t) > c$. Next, pick $a_1 \in Z$ such that for $t < -M + 1$,
$\tau_{a_1} q(t) < \delta(h)/2$. Thus, we have $\tau_{a_h} h > \tau_{a_1} q$. Let $k_1 \in \mathbb{N}$ be so large that for
$t < -k_1$, $\tau_{a_h} h(t) > c$ and for $t > k_1$, $\tau_{a_1} q(t) < 1 - \delta(h)/2$. For any $k > \max\{M, k_1\}$,
let

$$v_k^A(t) := \min\{\tau_{a_h} h(t), c\} \text{ for } t \in [-k,k].$$

Notice that since $k > k_1$, we have $v_k^A(t) = c$ for $t = \pm k$, and so we can extend $v_k^A$
to all of $\mathbb{R}$ by requiring $v_k^A$ to be $2k$-periodic. Notice that for $-M - 1 < t < k$,
we must have $v_k^A(t) = c$, since on this set $\tau_{a_h} h(t) > c$ because of the choice of $a_h$.

We want to use the flow $\varphi_s(\cdot)$ to get a candidate for a $w$ as in Condition (MB),
and $v_k^A$ is a good candidate for a initial value. However, it will be more
advantageous to use an initial condition which crosses 1/2 at most twice. If $h$
crosses 1/2 only twice, then so too does $v_k^A$, and we could use it as our initial
condition. However, we do not assume that $h$ crosses 1/2 exactly twice, and so we
need to modify $v_k^A$.

Let

$$[\alpha, \beta] := \text{closed convex hull of } \{ t \in \mathbb{R} \mid \tau_{a_h} h(t) < 1/2 \}.$$
Notice that we must have $-k < \alpha < \beta < -M - 1$ and $\tau_{a_1} h(\alpha) = 1/2 = \tau_{a_1} h(\beta)$. Let $\psi \in C_0^\infty(\alpha, \beta)$ be such that $0 < \psi(t) < \|\psi\|_{L^\infty}$ for all $t \in (\alpha, \beta)$. For $t \in [-k, k]$, we define

$$w_k^A(t) := \begin{cases} v_k^A(t) & \text{for } t \notin [\alpha, \beta] \\ \min\{1/2 - \psi(t), v_k^A(t)\} & \text{for } t \in [\alpha, \beta]. \end{cases}$$

Notice that $w_k^A(t) = v_k^A(t)$ for $t \in [-k, k] \setminus [\alpha, \beta]$, and thus we may extend $w_k^A$ to all of $\mathbb{R}$ by making $w_k^A$ 2k-periodic, and so $w_k^A \in E_k$. For $\|\psi\|_{L^\infty}$ sufficiently small, $w_k^A(t) > \delta(h)/2$ for $t \in [\alpha, \beta]$, and so $w_k^A > \tau_{a_1} q$. Furthermore, because $w_k^A \leq v_k^A$, $w_k^A \leq \tau_{a_1} h$. Because we assume that $\psi(t) > 0$ in the interior of $[\alpha, \beta]$, we know that $w_k^A$ crosses 1/2 at most twice in $[-k, k]$. In fact, since $h$ must intersect 1/2 and $w_k^A \leq \tau_{a_1} h$, we know that $w_k^A$ crosses 1/2 exactly twice.

Next, we claim that $w_k^A > q_k^A$. Indeed, if $-M \leq t \leq M$, we know that (since $\beta < -M - 1$) $w_k^A(t) = v_k^A(t) = c > \eta \geq q_k^A(t)$. In fact, this argument shows that $w_k^A > q_l^A$ for all $l \in \mathbb{N}$, since on the only interval in $[-lk, lk]$ where $q_l^A(t) \neq 0$, we will have $w_k^A(t) = c > \eta \geq q_l^A(t)$. Moreover, we must have $w_k^A(t) > \tau_{a_1} q(t)$, since $c > q(t)$ and $\tau_{a_1} h(t) > \tau_{a_1} q(t)$ for all $t \in \mathbb{R}$ by our choice of $a_1$. Thus, $w_k^A$ has the following four properties:

(i) $w_k^A > q_l^A$ for all $l \in \mathbb{N}$

(ii) $w_k^A < \tau_{a_1} h$ \hspace{1cm} (4.95)

(iii) $w_k^A > \tau_{a_1} q$.

(iv) $w_k^A$ intersects 1/2 exactly twice in $[-k, k]$.

Let $s_i$ be a sequence with $s_i \to \infty$ as $i \to \infty$ and for which $\varphi_{s_i}(q_k^A) \to q_k$ for all $k$. On a subsequence of $s_i$ we must then have $\varphi_{s_i}(w_k^A) \to w_k$ in $E_k$ for $w_k$ a
solution of (HS). Because of (i) - (iv) in (4.95), \( w_k \) must satisfy

\[
\begin{align*}
\text{(i)} & \quad w_k > q_{lk} \text{ for all } l \geq 2, \ l \in \mathbb{N} \\
\text{(ii)} & \quad w_k < \tau_{ah} h \\
\text{(iii)} & \quad w_k > \tau_{a1} q \\
\text{(iv)} & \quad w_k \text{ intersects } 1/2 \text{ exactly twice in } [-k, k].
\end{align*}
\]

(i)-(iii) follow from ODE uniqueness, and the fact that \( w_k^A \) is \( 2k \)-periodic, while \( q_{lk} \) is \( 2lk \) periodic, and \( h, q \) are not periodic. To see why (4.96)(iv) holds, note that by Proposition 4.2.8, \( w_k \) intersects 1/2 at most in \([-k, k]\). But \( \tau_{a1} q(\tau_(\hat{t})) < w_k(\hat{t}) < \tau_{ah} h(\hat{t}) \). Because of the structure of \( q, h \), there is a \( t_q, t_h \) such that \( 1/2 < \tau_{a1} q(t_q) < w_k(t_q) \) and \( 1/2 > \tau_{ah} h(t_h) > w_k(t_h) \). Thus, \( w_k \) must intersect 1/2 exactly twice in \([-k, k]\).

We claim that \( w_k \) satisfies Condition (MB). By (4.96)(i), we need only find an interval \((a, b)\) such that \( \Gamma_{lk} \subset (a, b) \). We know that \( w_k(\alpha) < \tau_{ah} h(\alpha) = 1/2 \) and similarly, \( w_k(\beta) < 1/2 \). Therefore, \( w_k(\beta) < 1/2 \) and \( w_k(\alpha + 2k) < 1/2 \). Since \( q \) crosses 1/2 exactly twice and \( k \) is sufficiently large that \( \tau_{a1} q(t) < \delta(h)/2 \) for all \( t > k \), there must be a \( \hat{t} \in (\alpha, \alpha + 2k) \) such that \( \tau_{a1} q(\hat{t}) > 1/2 \). But then \( w_k(\hat{t}) > 1/2 \). Let \((a, b)\) be the maximal interval containing \( \hat{t} \) such that \( w_k(t) > 1/2 \) for all \( t \in (a, b) \).

Notice that by (4.95)(i)-(ii), for all \( l \), we have \( q_{lk}^A < \tau_{ah} h \), hence \( q_{lk} < \tau_{ah} h \). Thus, by the same argument as above, we know that \( q_{lk}(\beta) < 1/2 \) and \( q_{lk}(\alpha + 2k) < 1/2 \). We claim that there must be a \( \hat{t}_l \in (\beta, \alpha + 2k) \) for which \( q_{lk}(\hat{t}_l) > 1/2 \). If not, then there must be a \( \hat{t}_l \in (\alpha, \beta) \) for which \( q_k(\hat{t}_l) > 1/2 \). However, \( q_{lk}(t) < 1/2 \) for all \( t \in (\alpha, \beta) \), since in this interval \( q_{lk}^A(t) = 0 < w_k^A(t) < 1/2 \). Thus, if
$q_{lk}(\tilde{t}_l) > 1/2$, there would be a first $i$ for which $\varphi_{s_i}(q^A_{lk})(\tilde{t}_l) = 1/2$. Since $q_{lk}$ crosses 1/2 exactly twice in any interval of length $2lk$, we must have $\varphi_{s_i}(q^A_{lk})$ tangent to 1/2 at $\tilde{t}_l$. In order not to create too many intersections with 1/2, we must have that $\varphi_{s_i}(q^A_{lk}) \leq 1/2$, and so $\varphi_{s_i}(\varphi_{s_i}(q^A_{lk})) < 1/2$ for $s > 0$ by the maximum principle. But this would contradict Proposition 4.2.10.

Therefore, there is a $\hat{t}_l \in (\beta, \alpha + 2k)$ such that $q_{lk}(\hat{t}_l) > 1/2$. But then, using $w_k > q_{lk}$ and the maximality of $(a, b)$, $\{t \in (\beta, \alpha + 2k) \mid q_{lk}(t) > 1/2\} \subset (a, b)$. Thus, Condition (MB) is satisfied.

**Remark:** (1) It is tempting to suggest that if there exist heteroclinics $h_1, h_2$ from 0 to 1 and from 1 to 0, then Condition (MB) is satisfied. However, we may have $\varphi_{s}(w^A_{k})$ intersecting $h_1, h_2$ exactly once for all $s > 0$, and $\varphi_{s}(w^A_{k}) \rightarrow 1$ as $s \rightarrow \infty$. The key thing about a homoclinic is that it provides a barrier to keep $\varphi_{s}(w^A_{k})$ from ever being greater than 1/2 everywhere.

(2) It seems likely that one could prove the existence of multi-bump type solutions in the presence of a homoclinic to 1 directly, without having to take a detour through subharmonics. Indeed, by considering appropriate translations of $q, h$, we pick an initial $w^A \in W^{1,2}(\mathbb{R})$ such that $w^A$ is above some set of translates of $q$ and below some set of translates of $h$. Then, taking $\varphi_{s}(w^A)(t)$ to be the unique solution of $u_s = u_{tt} + V_q(t, u)$ with $u(0, t) = w^A(t)$ and noting that $\varphi_{s}(w^A) \in W^{1,2}(\mathbb{R})$ for all $s \geq 0$, we would like to say that $\varphi_{s_i}(w^A)$ converges to some solution of (HS) that is above the translates of $q$ and below the translates of $h$. In particular, if

$$I(w) := \int_{\mathbb{R}} \left( \frac{1}{2} \dot{w}^2(t) - V(t, w(t)) \right) \, dt,$$

then $I$ is not bounded from below for $q \in W^{1,2}(\mathbb{R})$. Thus, $I$ does not satisfy the
(PS) condition. However, if we could show that \( \{ t \in \mathbb{R} \mid \varphi_{s_i}(w^A)(t) \geq 1/2 \} \) is bounded independently of \( i \), then \( I(\varphi_{s_i}(u)) \) is bounded from below, and we can argue as in the periodic case (plus some extra technical arguments) to show there is a sequence \( s_i \to \infty \) as \( i \to \infty \) for which \( \varphi_{s_i}(w^A) \) is a (PS) sequence for \( I \). Because of the boundedness of \( \{ t \in \mathbb{R} \mid \varphi_{s_i}(w^A)(t) \geq 1/2 \} \), we can find a subsequence \( s_{ij} \) for which \( \varphi_{s_{ij}}(w^A) \) converges weakly in \( W^{1,2}(\mathbb{R}) \) and strongly in \( L^\infty_{loc}(\mathbb{R}) \) to some \( w \in W^{1,2}(\mathbb{R}) \). Then, \( w \) will have the same order properties as \( w^A \). Moreover, since \( w \) is the weak limit of a (PS) sequence, \( w \) will satisfy (HS). The point of the proof above is to show that the existence of a homoclinic to 1 implies Condition (MB). However, it seems unlikely that Condition (MB) implies the existence of a homoclinic to 1. Unfortunately, we have been unable to find an example of when Condition (MB) is satisfied and there are no homoclinics to 1.

Thus to provide an example, it suffices to provide an example for which there is a homoclinic to 1. Suppose that \( \tilde{V} \in C^3(\mathbb{R}, \mathbb{R}) \) satisfies:

\[
\begin{align*}
(1) \quad & \tilde{V}(0) > \tilde{V}(t) \text{ for all } t \neq 0 \\
(2) \quad & \tilde{V}(-1) \text{ is a local maximum for } \tilde{V} \\
(3) \quad & \tilde{V}(-1/2) \text{ is a local minimum} \\
(4) \quad & \tilde{V}_q(x) \neq 0 \text{ when } x \notin \{0, -1/2, -1\}.
\end{align*}
\]

Then, if \( a(t) > 0 \) is \( T \)-periodic, Coti-Zelati and Rabinowitz have shown in [7] by a gluing argument that for \( T \) sufficiently large and \( \bar{a} - a = \sup a - \inf a \) small, there is a solution \( \tilde{h} \) of

\[
\ddot{q}(t) = -a(t)\tilde{V}_q(q(t)) = -V^*_q(t, q(t))
\]
homoclinic to 0, where $V^*(t, q) := -a(t)\tilde{V}(q)$.

**Lemma 4.5.2.** If $\tilde{V}$ satisfies (1)-(4) and $a > 0$, then $-1 \leq \tilde{h}(t) \leq 0$ for all $t$.

*Proof.* Because of the asymptotics of $\tilde{h}$, there is a $t_1$ with $\tilde{h}(t_1) = \inf \tilde{h}(t)$. Then, $\dot{\tilde{h}}(t_1) = 0$. If $\tilde{h}(t_1) < -1$, then (2) and (4) imply that $\ddot{\tilde{h}}(t_1) = -a(t_1)\tilde{V}_q(t_1, \tilde{h}(t_1)) < 0$. Then, for all $t$ close to $t_1$, $\dot{\tilde{h}}(t)$ is decreasing. Since $\dot{\tilde{h}}(t_1) = 0$, for all $t > t_1$ that are close to $t_1$ we have $\dot{\tilde{h}}(t) < 0$. Thus, $\tilde{h}$ is decreasing close to $t_1$, which is impossible.

If there is a $t_2$ with $\tilde{h}(t_2) > 0$, then because of the asymptotics of $\tilde{h}$, there is a $t_3$ with $0 < \sup \tilde{h}(t) = \tilde{h}(t_3)$. Then $\dot{\tilde{h}}(t_3) = 0$, and by (1) and (4) we have $\ddot{\tilde{h}}(t_3) = -a(t_3)\tilde{V}_q(\tilde{h}(t_3)) > 0$. But then arguing as above, we must have $\tilde{h}$ increasing at $t_3$, which is impossible. \hfill $\square$

Now, we change variables to take $V^*(t, q)$ to a potential $V(t, q)$ that satisfies (DV1)-(DV6). In fact, with (1)-(4), we will satisfy (DV5)’. Let

$$V(t, q) := a(t) \left( \tilde{V}(q - 1) - \tilde{V}(-1) \right).$$

Then, notice that $V_q(t, q) = a(t)\tilde{V}_q(q - 1)$, so the only zeros of $V_q(t, q)$ are when $q - 1 \in \{-1, -1/2, 0\}$, so when $q \in \{0, 1/2, 1\}$. Thus,

$$V(t, 1) = a(t) \left( \tilde{V}(0) - \tilde{V}(-1) \right) \geq a \left( \tilde{V}(0) - \tilde{V}(-1) \right) > 0 \text{ by (1)}$$

$$V(t, 1/2) = a(t) \left( \tilde{V}(-1/2) - \tilde{V}(-1) \right) \leq \tilde{a} \left( \tilde{V}(-1/2) - \tilde{V}(-1) \right) < 0$$

$$V(t, 0) = a(t) \left( \tilde{V}(-1) - \tilde{V}(-1) \right) \equiv 0$$

Moreover, because $a > 0$, $V_q(t, q)$ has the same sign as $\tilde{V}_q(q)$. Hence $q = 1$ is a global maximum for every $t$, $q = 1/2$ is a local minimum for all $t$ and $q = 0$ is a
local maximum for all $t$. By modifying $V$ suitably for $q \notin [0, 1]$, we will have the growth conditions. Finally, let $h(t) := \tilde{h}(t) + 1$. Notice that $h$ is homoclinic to 0, and

$$
\ddot{h}(t) = \ddot{\tilde{h}}(t)
= -a(t)\tilde{V}_{q}(\tilde{h}(t))
= -a(t)\tilde{V}_{q}(\tilde{h}(t) + 1 - 1)
= -a(t)\tilde{V}_{q}(h(t) - 1)
= -V_{q}(t, h(t)),
$$

so $h$ solves (HS). In addition, $0 \leq h \leq 1$, and so by ODE uniqueness $0 < h < 1$. Finally, note that as defined, $V$ is $T$-periodic in $t$. However, the results of the preceding sections apply equally as well to a potential that is $T$-periodic.
Appendix A

A.1 Semi-linear Parabolic Differential Equations

In this appendix, we prove the various results that we used earlier about the solutions of the following parabolic partial differential equation:

\[(PDE) \quad w_s(s, t) = w_{tt}(s, t) + V_q(t, w(s, t)) \quad w(0, t) = u(t)\]

In particular, we want to prove

**Theorem A.1.1.** (1) For any \(u \in E_k\), there is a unique solution \(w(s, t)\) of (PDE) such that \(s \mapsto w(s, \cdot)\) is in \(C([0, \infty), E_k)\). Moreover, \(w\) is a classical solution, in the sense that (a) for every \(t \in \mathbb{R}\), \(s \mapsto w(s, t)\) is differentiable for every \(s > 0\), (b) for every \(s > 0\), \(t \mapsto w(s, t)\) is twice differentiable, and (c) (PDE) is satisfied pointwise.

(2) For \(u \in E_k\), let \(\varphi^k_s(u)(t) := w(s, t)\), where \(w\) satisfies (PDE), with initial condition \(u\). This satisfies the semi-group property: \(\varphi^k_{s_1+s_2}(u) = \varphi^k_{s_1}(\varphi^k_{s_2}(u))\) for any \(s_1, s_2 > 0\).

(3) \(\varphi^k_s(u)\) is continuous in \(u\): for any \(\varepsilon > 0\) and \(T > 0\), there is a \(\delta = \delta(\varepsilon, T, u) > 0\) such that if \(\|u - v\|_{E_k} < \delta\), then \(\|\varphi^k_s(u) - \varphi^k_s(v)\|_{E_k} < \varepsilon\) for all \(0 \leq s \leq T\).

(4) For any sequence \(s_i \to \infty\) and any \(u \in E_k\), there is a subsequence \(s_{i_j}\) and a solution \(\tilde{u} \in E_k\) of (HS) such that \(\|\varphi^k_{s_{i_j}}(u) - \tilde{u}\|_{E_k} \to 0\).
To prove Theorem A.1.1, we first consider the case where the initial condition $u$ is a bounded uniformly continuous function. This contains the cases we are interested in: $u \in W^{1,2}(\mathbb{R})$ and $u \in E_k$, where $E_k$ is the space of $2k$-periodic $W^{1,2}$ functions, since if $u$ is any of these spaces, $u$ will be bounded and uniformly continuous.

Throughout, $\| \cdot \|_{\infty}$ will be the $L^\infty$ norm on $\mathbb{R}$. We define

**Definition A.1.2.**

$$BUC^n(\mathbb{R}) := \{ u \in C^n(\mathbb{R}) \mid \| D^m u \|_{\infty} < \infty \text{ and } D^m u \text{ is uniformly continuous for all } m \leq n \}$$

Notice that $(BUC(\mathbb{R}), \| \cdot \|_{\infty})$ is a Banach space.

**Definition A.1.3.** $X := C([0,T], BUC(\mathbb{R}))$, and if $v \in X$, we put $\| v \|_* := \sup_{s \in [0,T]} \| v(s, \cdot) \|_{\infty}$.

Notice that $(X, \| \cdot \|_*)$ is also a Banach space. For future reference, we prove

**Lemma A.1.4.** If $v \in X$, then the family of functions $t \mapsto v(\alpha, t)$ for $\alpha \in [0,T]$ is equicontinuous.

**Proof.** Let $\varepsilon > 0$ be given. Since $v([0,T], \cdot) \subset BUC(\mathbb{R})$ is compact, there is a finite collection $a_1, \ldots, a_n$ such that $v([0,T], \cdot) \subset \bigcup B_{\varepsilon/3}(v(a_i, \cdot))$. Since each $v(a_i, \cdot)$ is uniformly continuous, there is an $\delta_i > 0$ such that if $|t_1 - t_2| < \delta_i$, then $|v(a_i, t_1) - v(a_i, t_2)| < \varepsilon/3$. Let $\delta := \min\{\delta_1, \ldots, \delta_n\}$. Suppose then that $|t_1 - t_2| < \delta$. Then, for any $a \in [0,T]$, $v(a, \cdot) \in B_{\varepsilon/3}(v(a_i, \cdot))$ for some $i$, and so

$$|v(a, t_1) - v(a, t_2)| \leq |v(a, t_1) - v(a_i, t_1)| + |v(a_i, t_1) - v(a_i, t_2)| + |v(a_i, t_2) - v(a, t_2)|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$
since \( \delta \leq \delta_i \) and \( v(a, \cdot) \in B_{\varepsilon/3}(v(a_i, \cdot)) \).

For \( u \in BUC(\mathbb{R}) \) and \( s > 0 \), we define

**Definition A.1.5.**

\[
e^{s\Delta}u(t) := \frac{1}{(4\pi s)^{1/2}} \int_{\mathbb{R}} e^{-\frac{r^2}{4s}} u(t-r) \, dr = \frac{1}{(4\pi s)^{1/2}} \int_{\mathbb{R}} e^{-(t-r)^2/4s} u(r) \, dr
\]

Thus, \( e^{s\Delta}u = K_s \ast u \), where \( K_s \) is the heat kernel. Notice that for \( s_1, s_2 > 0 \), we have \( e^{s_1\Delta}(e^{s_2\Delta}u)(t) = e^{(s_1+s_2)\Delta}u(t) \), and \( e^{s\Delta}u(t) \) is the unique solution of the heat equation with initial value \( u \). (For example, see [12] or [9].) We now collect some lemmas about \( e^{s\Delta}u \) when \( u \in BUC(\mathbb{R}) \).

**Lemma A.1.6.** \( e^{s\Delta} : BUC(\mathbb{R}) \to BUC(\mathbb{R}) \) is linear, and \( \|e^{s\Delta}u\|_{\infty} \leq \|u\|_{\infty} \).

**Proof.** The linearity is clear. We have

\[
|e^{s\Delta}u(t)| \leq \frac{1}{(4\pi s)^{1/2}} \int_{\mathbb{R}} e^{-\frac{r^2}{4s}} |u(t-r)| \, dr = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2} |u(t-2s^{1/2}y)| \, dy
\]

after making the change of variable \( y = \frac{r}{\sqrt{4s}} \). But then we have

\[
|e^{s\Delta}u(t)| \leq \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2} \|u\|_{\infty} \, dy.
\]

To see that \( e^{s\Delta}u \in BUC(\mathbb{R}) \), notice that

\[
|e^{s\Delta}u(t_1) - e^{s\Delta}u(t_2)| \leq \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2} |u(t_1-2s^{1/2}y) - u(t_2-2s^{1/2}y)| \, dy \quad (A.1)
\]

But \( u \in BUC(\mathbb{R}) \), hence for any \( \varepsilon > 0 \), there is a \( \delta = \delta(\varepsilon) > 0 \) such that if \( |t_1 - t_2| < \delta(\varepsilon) \), then \( |u(t_1) - u(t_2)| < \varepsilon \). But then, if \( |t_1 - t_2| < \delta(\varepsilon) \), we have

\[
|(t_1-2s^{1/2}y) - (t_2-2s^{1/2}y)| < \delta(\varepsilon) \text{ for all } y \in \mathbb{R}, \text{ and so by (A.1), we have}
\]

\[
|e^{s\Delta}u(t_1) - e^{s\Delta}u(t_2)| < \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2} \varepsilon \, dy = \varepsilon \quad (A.2)
\]

whenever \( |t_1 - t_2| < \delta(\varepsilon) \).
The next lemma concerns how $e^{s\Delta}u$ assumes its initial conditions.

**Lemma A.1.7.** $\|e^{s\Delta}u - u\|_\infty \to 0$ as $s \searrow 0$.

**Proof.** Notice that since $\int_{\mathbb{R}} K_s(r)dr = 1$ for all $s > 0$, we have

$$e^{s\Delta}u(t) - u(t) = \frac{1}{\sqrt{4\pi s}} \int_{\mathbb{R}} e^{-\frac{r^2}{4s}} (u(t) - u(t)) dr.$$  \hfill (A.3)

Let $\varepsilon > 0$ be given. We need to show that there is an $\alpha > 0$ such that if $0 < s < \alpha$, then

$$\|e^{s\Delta}u - u\|_\infty < \varepsilon.$$ \hfill (A.4)

We know that

$$e^{s\Delta}u(t) - u(t) = \frac{1}{\sqrt{4\pi s}} \int_{\mathbb{R}} e^{-\frac{r^2}{4s}} (u(t) - u(t)) dr$$

$$= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2} (u(t - 2s^{1/2}y) - u(t)) dy$$  \hfill (A.5)

$$= \frac{1}{\sqrt{\pi}} \int_{|y| \geq \eta} e^{-y^2} (u(t - 2s^{1/2}y) - u(t)) dy$$

$$+ \frac{1}{\sqrt{\pi}} \int_{|y| < \eta} e^{-y^2} (u(t - 2s^{1/2}y) - u(t)) dy$$

Thus, from (A.5), we have

$$|e^{s\Delta}u(t) - u(t)| \leq 2\|u\|_\infty \frac{1}{\sqrt{\pi}} \int_{|y| \geq \eta} e^{-y^2} dy + \frac{1}{\sqrt{\pi}} \int_{|y| < \eta} e^{-y^2} |u(t - 2s^{1/2}y) - u(t)| dy$$  \hfill (A.6)

Now, pick $\eta = \eta(\varepsilon) > 0$ so large that

$$2\|u\|_\infty \frac{1}{\sqrt{\pi}} \int_{|y| \geq \eta} e^{-y^2} dy < \varepsilon.$$ \hfill (A.7)

Because $u$ is uniformly continuous, there is a $\delta = \delta(\varepsilon)$ such that for any $t_1, t_2 \in \mathbb{R}$ with $|t_1 - t_2| < \delta(\varepsilon)$, we have $|u(t_1) - u(t_2)| < \varepsilon$. Let $\alpha(\varepsilon) := \frac{\delta(\varepsilon)^2}{4\eta(\varepsilon)}$. Notice that $\alpha(\varepsilon)$
is independent of \( t \). Suppose that \( 0 < s < \alpha(\varepsilon) \). Then, we have \( 2s^{1/2}\eta(\varepsilon) < \delta(\varepsilon) \), and so for all \( y \) with \( |y| < \eta(\varepsilon) \), \( |2s^{1/2}y| < \delta(\varepsilon) \), hence
\[
\frac{1}{\sqrt{\pi}} \int_{|y| < \eta(\varepsilon)} e^{-y^2} |u(t - 2s^{1/2}y) - u(t)| \, dy < \frac{\varepsilon}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2} \, dy = \varepsilon 
\tag{A.8}
\]
for all \( t \in \mathbb{R} \). Combining (A.6), (A.7) and (A.8), we have
\[
|e^{s\Delta} u(t) - u(t)| < 2\varepsilon
\]
for every \( t \in \mathbb{R} \), and thus \( \|e^{s\Delta} u - u\|_{\infty} < 2\varepsilon \) when \( 0 < s < \alpha(\varepsilon) \). \( \square \)

**Corollary A.1.8.** Let \( K \subset BUC(\mathbb{R}) \) be compact. Then,
\[
\sup_{u \in K} \|e^{s\Delta} u - u\|_{\infty} \to 0
\]
as \( s \searrow 0 \).

**Proof.** Let \( \varepsilon > 0 \) be given. Then, there is a finite subset \( u_1, \ldots, u_n \) such that \( K \subset \bigcup B_{\varepsilon}(u_i) \). For each \( u_i \), Lemma A.1.7 implies that there is an \( \alpha_i > 0 \) such that \( \|e^{s\Delta} u_i - u_i\|_{\infty} < \varepsilon \) whenever \( 0 < s < \alpha_i \). Let \( \alpha := \min\{\alpha_1, \ldots, \alpha_n\} > 0 \). Suppose that \( 0 < s < \alpha \), and let \( u \in K \). Then, \( u \in B_{\varepsilon}(u_i) \) for some \( i \), and so using Lemma A.1.6, we have
\[
\|e^{s\Delta} u - u\|_{\infty} \leq \|e^{s\Delta} u - e^{s\Delta} u_i\|_{\infty} + \|e^{s\Delta} u_i - u_i\|_{\infty} + \|u_i - u\|_{\infty}
\]
\[
\leq 2\|u_i - u\|_{\infty} + \|u_i - u\|_{\infty}
\]
\[
< 3\varepsilon
\]
since \( u \in B_{\varepsilon}(u_i) \) and \( 0 < s < \alpha \leq \alpha_i \). \( \square \)

Next, we collect some important lemmas about the differentiability properties of \( e^{s\Delta} u \).
Lemma A.1.9. For $s > 0$, $t \mapsto e^{s\Delta}u(t)$ is differentiable, and there is a constant $C_1$ such that

$$\left\| \frac{\partial}{\partial t} e^{s\Delta}u \right\|_\infty \leq \frac{C_1}{\sqrt{s}} \|u\|_\infty.$$ 

Proof. We claim that in fact

$$\frac{\partial}{\partial t} e^{s\Delta}u(t) = \frac{1}{\sqrt{4\pi s}} \int_{\mathbb{R}} -(t-r) e^{-\frac{(t-r)^2}{4s}} u(r) dr. \quad (A.9)$$

Assuming (A.9), we then have

$$\frac{\partial}{\partial t} e^{s\Delta}u(t) = \frac{1}{\sqrt{\pi s}} \int_{\mathbb{R}} -ye^{-y^2} u(t - 2s^{1/2}y) dy,$$

and so

$$\left| \frac{\partial}{\partial t} e^{s\Delta}u(t) \right| \leq \frac{1}{\sqrt{s}} \left( \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} |y| e^{-y^2} dy \right) \|u\|_\infty.$$ 

The proof of (A.9) follows from standard theorems in real analysis, see for example [24].

Next, we show that if $u$ is differentiable, then $s \mapsto e^{s\Delta}u(t)$ is differentiable for $s > 0$.

Lemma A.1.10. If $u, \dot{u} \in BUC(\mathbb{R})$, then for all $s > 0$ and any $t \in \mathbb{R}$, $s \mapsto e^{s\Delta}u(t)$ is differentiable, and there is a constant $C_2$ such that

$$\left\| \frac{\partial}{\partial s} e^{s\Delta}u \right\|_\infty \leq \frac{C_2}{\sqrt{s}} \|\dot{u}\|_\infty.$$ 

Proof. We have

$$e^{s\Delta}u(t) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2} u(t - 2s^{1/2}y) dy.$$
By standard theorems, we can differentiate in $s$, since $u$ is differentiable, and $\dot{u}$ is bounded. In particular,

\[
\frac{\partial}{\partial s} e^{s\Delta} u(t) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2} \dot{u}(t - 2s^{1/2}y)(-ys^{-1/2}) dy
\]

\[
= \frac{1}{\sqrt{\pi} s} \int_{\mathbb{R}} -ye^{-y^2} \dot{u}(t - 2s^{1/2}) dy,
\]

and so

\[
\left\| \frac{\partial}{\partial s} e^{s\Delta} u \right\|_{\infty} \leq \frac{1}{\sqrt{s}} \left( \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} |y| e^{-y^2} dy \right) \left\| \dot{u} \right\|_{\infty}
\]

\[\square\]

**Proposition A.1.11.** Suppose that $v \in X$. Then

(i) $s \mapsto \int_0^s e^{(s-a)\Delta} v(a, t) da \in C([0, T], BUC^1(\mathbb{R}))$

(ii) $\frac{\partial}{\partial t} \int_0^s e^{(s-a)\Delta} v(a, t) da = \int_0^s \frac{\partial}{\partial t} e^{(s-a)\Delta} v(a, t) da$

(iii) $\left\| \frac{\partial}{\partial t} \int_0^s e^{(s-a)\Delta} v(a, t) da \right\|_{\infty} \leq C \sqrt{s} \|v\|_s$

**Proof.** We first show that $t \mapsto \int_0^s e^{(s-a)\Delta} v(a, t) da \in BUC(\mathbb{R})$ for all $s \geq 0$. This is clear for $s = 0$. Fix an $s > 0$. For each $t \in \mathbb{R}$, $a \mapsto e^{(s-a)\Delta} v(a, t)$ is continuous for $a \in [0, s]$. Thus, the integral makes sense, and we can define $\int_0^s e^{(s-a)\Delta} v(a, t) da$ pointwise. Next, note that

\[
e^{(s-a)\Delta} v(a, t_1) - e^{(s-a)\Delta} v(a, t_2) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2} \left( v(a, t_1 - 2(s - a)^{1/2}y) - v(a, t_2 - 2(s - a)^{1/2}y) \right) dy.
\]

(A.10)

Let $\varepsilon > 0$ be given. By Lemma A.1.4, there is a $\delta = \delta(\varepsilon) > 0$ such that if $|\hat{t}_1 - \hat{t}_2| < \delta(\varepsilon)$, then $|v(a, \hat{t}_1) - v(a, \hat{t}_2)| < \varepsilon$ for all $a \in [0, T]$. Thus, if $|t_1 - t_2| < \delta(\varepsilon)$,
(A.10) implies that for all $a \in [0, s]$

$$|e^{(s-a)\Delta}v(a, t_1) - e^{(s-a)\Delta}v(a, t_2)|$$

$$\leq \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2} |v(a, t_1 - 2(s-a)^{1/2}y) - v(a, t_2 - 2(s-a)^{1/2}y)| dy$$

(A.11)

$$< \varepsilon \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2} dy = \varepsilon.$$

Thus,

$$\left| \int_0^s e^{(s-a)\Delta}v(a, t_1) da - \int_0^s e^{(s-a)\Delta}v(a, t_2) da \right| < s \varepsilon \leq T \varepsilon$$

whenever $|t_1 - t_2| < \delta(\varepsilon)$. Thus, $t \mapsto \int_0^s e^{(s-a)\Delta}v(a, t) da$ is uniformly continuous.

Moreover, by Lemma A.1.6

$$\left| \int_0^s e^{(s-a)\Delta}v(a, t) da \right| \leq \int_0^s |e^{(s-a)\Delta}v(a, t)| da$$

$$\leq \int_0^s \|v(a, \cdot)\|_{\infty} da$$

$$\leq T \|v\|_{\ast}$$

so $t \mapsto \int_0^s e^{(s-a)\Delta}v(a, t) da \in BUC(\mathbb{R})$.

Next, we show that if $s_n \to s$ as $n \to \infty$, then

$$\left\| \int_0^{s_n} e^{(s-a)\Delta}v(a, \cdot) da - \int_0^s e^{(s-a)\Delta}v(a, \cdot) da \right\|_{\infty} \to 0 \quad (A.12)$$

as $n \to \infty$.

We consider two cases: (a) $s_n \geq s$ for all $n$, and (b) $s_n \leq s$ for all $n$. Once we have shown that (A.12) holds for these cases, (A.12) holds for any sequence
Suppose now that (a) holds. Then

\[
\int_0^{s_n} e^{(s_n-a)\Delta} v(a, t) da - \int_s^{s_n} e^{(s_n-a)\Delta} v(a, t) da = \int_s^s e^{(s_n-a)\Delta} v(a, t) da - e^{(s-a)\Delta} v(a, t) da + \int_s^{s_n} e^{(s_n-a)\Delta} v(a, t) da
\]

(A.13)

\[
\int_0^{s_n} e^{(s_n-a)\Delta} v(a, t) da - \int_0^{s_n} e^{(s-a)\Delta} v(a, t) da =: A_n(t) + B_n(t).
\]

But by Lemma A.1.6, we know that

\[
|B_n(t)| \leq \int_s^{s_n} |e^{(s_n-a)\Delta} v(a, t)| da \leq \int_s^{s_n} \|v\|^* da
\]

(A.14)

\[
\leq |s_n - s\|v\|^* \rightarrow 0
\]

as \(n \rightarrow \infty\). Because of (A.13) and (A.14), to show (A.12), we need to show that \(\|A_n\|_\infty \rightarrow 0\) as \(n \rightarrow \infty\). From (A.13), we have

\[
|A_n(t)| = \left| \int_0^s \left( e^{(s_n-a)\Delta} v(a, t) - e^{(s-a)\Delta} v(a, t) \right) da \right|
\]

(A.15)

\[
\leq \int_0^s \|e^{(s_n-a)\Delta} v(a, \cdot) - v(a, \cdot)\|_\infty da
\]

\[
\leq s \sup_{a \in [0,s]} \|e^{(s-a)\Delta} v(a, \cdot) - v(a, \cdot)\|_\infty.
\]

Since (A.15) is uniform in \(t\), we have that

\[
\|A_n\|_\infty \leq s \sup_{a \in [0,s]} \|e^{(s-a)\Delta} v(a, \cdot) - v(a, \cdot)\|_\infty
\]

(A.16)

as \(n \rightarrow \infty\). Now, Corollary A.1.8 implies that (A.16) goes to 0 as \(n \rightarrow \infty\). Case (b) where \(s_n \leq s\) for all \(n\) is handled similarly. Thus,
defines an element of $X$.

Next, we show that for any $s \geq 0$,

$$t \mapsto \int_0^s e^{(s-a)\Delta}v(a,t)da \quad (A.17)$$

is differentiable. For $s = 0$ this is clear, and in fact the derivative is identically 0. Fix an $s > 0$ and a $t \in \mathbb{R}$. By Lemma A.1.9, for every $a \in [0, s)$, $t \mapsto e^{(s-a)\Delta}v(a,t)$ is differentiable, and

$$\left\| \frac{\partial}{\partial t} e^{(s-a)\Delta}v(a, \cdot) \right\|_\infty \leq \frac{C_1}{\sqrt{s-a}} \left\| v(a, \cdot) \right\|_\infty. \quad (A.18)$$

Thus, for every $a \in [0, s)$,

$$\frac{1}{h} \left( e^{(s-a)\Delta}v(a, t+h) - e^{(s-a)\Delta}v(a, t) \right) \rightarrow \frac{\partial}{\partial t} (e^{(s-a)\Delta}v(a, t)) \quad (A.19)$$
as $h \to 0$. If $h > 0$, then (A.18) implies that

$$\left| \frac{1}{h} \left( e^{(s-a)\Delta}v(a, t+h) - e^{(s-a)\Delta}v(a, t) \right) \right| = \left| \frac{1}{h} \int_t^{t+h} \frac{d}{dx} \left( e^{(s-a)\Delta}v(a, x) \right) dx \right|$$

$$\leq \frac{1}{h} \int_t^{t+h} \left| \frac{d}{dx} e^{(s-a)\Delta}v(a, x) \right| dx$$

$$\leq \frac{1}{h} \int_t^{t+h} \left\| \frac{\partial}{\partial t} e^{(s-a)\Delta}v(a, \cdot) \right\|_\infty dx$$

$$\leq \frac{C_1}{\sqrt{s-a}} \left\| v(a, \cdot) \right\|_\infty \leq \frac{C_1}{\sqrt{s-a}} \| v \|_*.$$  

A similar estimate holds for $h < 0$, and so we have shown that

$$\left| \frac{1}{h} \left( e^{(s-a)\Delta}v(a, t+h) - e^{(s-a)\Delta}v(a, t) \right) \right| \leq \frac{C_1}{\sqrt{s-a}} \| v \|_* \quad (A.20)$$

Since $\int_0^s \frac{1}{\sqrt{s-a}} da < \infty$, (A.19), (A.20) and the dominated convergence theorem imply that

$$\int_0^s \frac{1}{h} \left( e^{(s-a)\Delta}v(a, t+h) - e^{(s-a)\Delta}v(a, t) \right) da \rightarrow \int_0^s \frac{\partial}{\partial t} e^{(s-a)\Delta}v(a, t)da$$
as \( h \to 0 \). Thus, (A.17) is differentiable, and the derivative is what we have claimed in (ii). Moreover, we know that
\[
\left| \int_0^s \frac{\partial}{\partial t} e^{(s-a)\Delta} v(a,t) \, da \right| \leq \int_0^s \left| \frac{\partial}{\partial t} e^{(s-a)\Delta} v(a,t) \right| \, da
\]
\[
\leq \int_0^s \frac{C_1}{\sqrt{s-a}} \|v\|^* \, da
\]
\[
= C_1 \sqrt{s} \|v\|^* ,
\]
which proves (iii). In addition, if \( s_n \searrow 0 \) as \( n \to \infty \), (A.21) implies that
\[
\left\| \frac{\partial}{\partial t} \int_0^s e^{(s-a)\Delta} v(a,\cdot) \, da \right\|_{\infty} \to 0
\]
as \( n \to \infty \). Therefore,
\[
s \mapsto \int_0^s \frac{\partial}{\partial t} e^{(s-a)\Delta} v(a,t) \, da
\]
is continuous as \( s = 0 \). To finish showing (i), we need to show that if \( s_n \to s \) as \( n \to \infty \) for \( s > 0 \), then
\[
\left\| \int_0^{s_n} \frac{\partial}{\partial t} e^{(s_n-a)\Delta} v(a,\cdot) \, da - \int_0^s \frac{\partial}{\partial t} e^{(s-a)\Delta} v(a,\cdot) \, da \right\|_{\infty} \to 0
\]
as \( n \to \infty \). We again consider two cases: (a) \( s_n \geq s \) for all \( n \) and (b) \( s_n \leq s \) for all \( n \). In case (a),
\[
\int_0^{s_n} \frac{\partial}{\partial t} e^{(s_n-a)\Delta} v(a,t) \, da - \int_0^s \frac{\partial}{\partial t} e^{(s-a)\Delta} v(a,t) \, da
\]
\[
= \int_0^s \left( \frac{\partial}{\partial t} e^{(s-a)\Delta} v(a,t) - \frac{\partial}{\partial t} e^{(s-a)\Delta} v(a,t) \right) \, da
\]
\[
+ \int_s^{s_n} \frac{\partial}{\partial t} e^{(s_n-a)\Delta} v(a,t) \, da
\]
\[
= A_n(t) + B_n(t)
\]
Therefore, to show (A.23), (A.24) implies that we need to show that \( \| A_n \|_\infty \to 0 \) and \( \| B_n \|_\infty \to 0 \) as \( n \to \infty \). We know that

\[
\begin{align*}
|B_n(t)| &\leq \int_s^{s_n} \left| \frac{\partial}{\partial t} e^{(s_n-a)\Delta} v(a,t) \right| da \\
&\leq \int_s^{s_n} \frac{C_1}{\sqrt{s_n-a}} \| v(a,\cdot) \|_\infty da \\
&\leq C_1 \| v \|^* \int_s^{s_n} \frac{1}{\sqrt{s_n-a}} da
\end{align*}
\]

(A.25)

for all \( t \in \mathbb{R} \). Thus, (A.25) implies that

\[
\| B_n \|_\infty \leq 2C_1 \| v \|^*(s_n - s)^{1/2} \to 0
\]

as \( n \to \infty \).

Turning to \( A_n \), we have

\[
A_n(t) = \int_0^t \left( \frac{\partial}{\partial t} e^{(s_n-a)\Delta} v(a,t) - \frac{\partial}{\partial t} e^{(s-a)\Delta} v(a,t) \right) da.
\]

(A.26)

Now,

\[
\begin{align*}
\frac{\partial}{\partial t} e^{(s_n-a)\Delta} v(a,t) - \frac{\partial}{\partial t} e^{(s-a)\Delta} v(a,t) \\
&= \frac{1}{\sqrt{\pi(s_n-a)}} \int_{\mathbb{R}} ye^{-y^2} v(a,t - 2(s_n-a)^{1/2}y) dy \\
&\quad - \frac{1}{\sqrt{\pi(s-a)}} \int_{\mathbb{R}} ye^{-y^2} v(a,t - 2(s-a)^{1/2}y) dy \\
&= \frac{1}{\sqrt{\pi(s_n-a)}} \int_{\mathbb{R}} ye^{-y^2} \left( v(a,t - 2(s_n-a)^{1/2}y) - v(a,t - 2(s-a)^{1/2}y) \right) dy \\
&\quad + \frac{1}{\sqrt{\pi}} \left( \frac{1}{\sqrt{s_n-a}} - \frac{1}{\sqrt{s-a}} \right) \int_{\mathbb{R}} ye^{-y^2} v(a,t - 2(s-a)^{1/2}y) dy.
\end{align*}
\]

(A.27)
Thus,

\[
\left| \frac{\partial}{\partial t} e^{(s_n - a)\Delta} v(a, t) - \frac{\partial}{\partial t} e^{(s - a)\Delta} v(a, t) \right| \\
\leq \frac{1}{\sqrt{\pi (s_n - a)}} \int_{\mathbb{R}} |y| e^{-y^2} \left| v(a, t - 2(s_n - a)^{1/2}y) - v(a, t - 2(s - a)^{1/2}y) \right| dy \\
+ \frac{1}{\sqrt{\pi}} \left| \frac{1}{\sqrt{s_n - a}} - \frac{1}{\sqrt{s - a}} \right| \int_{\mathbb{R}} |y| e^{-y^2} \|v(a, \cdot)\|_\infty dy,
\]

(A.28)

and so

\[
|A_n(t)| \leq \int_0^s \left| \frac{\partial}{\partial t} e^{(s_n - a)\Delta} v(a, t) - \frac{\partial}{\partial t} e^{(s - a)\Delta} v(a, t) \right| da \\
\leq \int_0^s \frac{1}{\sqrt{s_n - a}} \left( \int_{\mathbb{R}} |y| e^{-y^2} \left| v(a, t - 2(s_n - a)^{1/2}y) - v(a, t - 2(s - a)^{1/2}y) \right| dy \right) da \\
+ \int_0^s \|v\|^* \left| \frac{1}{\sqrt{s_n - a}} - \frac{1}{\sqrt{s - a}} \right| \left( \int_{\mathbb{R}} |y| e^{-y^2} dy \right) da
\]

(A.29)

Notice that

\[
\int_0^s \left| \frac{1}{\sqrt{s_n - a}} - \frac{1}{\sqrt{s - a}} \right| da = \int_0^s \frac{1}{\sqrt{s - a}} - \frac{1}{\sqrt{s_n - a}} da \\
= 2s^{1/2} + 2(s_n - s)^{1/2} - 2s_n^{1/2} \to 0
\]
as \(n \to \infty\). Thus, the last integral in (A.29) goes to 0 uniformly in \(t\) as \(n \to \infty\).

It remains only to show that

\[
\int_0^s \frac{1}{\sqrt{\pi (s_n - a)}} \left( \int_{\mathbb{R}} |y| e^{-y^2} \left| v(a, t - 2(s_n - a)^{1/2}y) - v(a, t - 2(s - a)^{1/2}y) \right| dy \right) da \\
\to 0
\]

(A.30)
as \( n \to 0 \) uniformly in \( t \in \mathbb{R}, a \in (0, s) \). Let \( \varepsilon > 0 \) be given. We claim that for all sufficiently large \( n \),

\[
\int_{\mathbb{R}} |y| e^{-y^2} |v(a, t - 2(s_n - a)^{1/2}y) - v(a, t - 2(s - a)^{1/2}y)| dy < \varepsilon
\]  

(A.31)

for any \( t \in \mathbb{R}, a \in (0, s) \). Assuming (A.31), we then have for all large \( n \)

\[
\int_0^s \frac{1}{\sqrt{\pi(s_n - a)}} \left( \int_{\mathbb{R}} |y| e^{-y^2} |v(a, t - 2(s_n - a)^{1/2}y) - v(a, t - 2(s - a)^{1/2}y)| dy \right) da 
\]

\[
< \varepsilon \int_0^s \frac{1}{\sqrt{\pi(s_n - a)}} da 
\]

\[
\leq \frac{\varepsilon}{\sqrt{\pi}} (2s_n^{1/2} - 2(s_n - s)^{1/2}) 
\]

(A.32)

\[
\leq C \varepsilon
\]

for all \( t \in \mathbb{R} \). Thus (A.32) implies (A.30).

To verify (A.31), we have

\[
\int_{\mathbb{R}} |y| e^{-y^2} |v(a, t - 2(s_n - a)^{1/2}y) - v(a, t - 2(s - a)^{1/2}y)| dy 
\]

\[
\leq 2\|v\| \int_{|y| \geq \eta} |y| e^{-y^2} dy 
\]

(A.33)

\[
+ \int_{|y| < \eta} |y| e^{-y^2} |v(a, t - 2(s_n - a)^{1/2}y) - v(a, t - 2(s - a)^{1/2}y)| dy 
\]

For any \( \varepsilon > 0 \), there is an \( \eta = \eta(\varepsilon) \) such that

\[
2\|v\| \int_{|y| \geq \eta} |y| e^{-y^2} dy < \varepsilon
\]  

(A.34)

Next, since \( v \in X \), Lemma A.1.4 implies there is a \( \delta = \delta(\varepsilon) > 0 \) such that if \( |\hat{t}_1 - \hat{t}_2| < \delta(\varepsilon) \), then \( |v(a, \hat{t}_1) - v(a, \hat{t}_2)| < \varepsilon \) for any \( a \in [0, T] \). Now, since \( g(x) := x^{1/2} \) is uniformly continuous on \([0, \infty)\), there is an \( \xi = \xi(\varepsilon, \eta) = \xi(\varepsilon) > 0 \) such that if \( |x_1 - x_2| < \xi(\varepsilon) \), then

\[
|x_1^{1/2} - x_2^{1/2}| < \frac{\delta(\varepsilon)}{2\eta(\varepsilon)}.
\]  

(A.35)
Let $N = N(\varepsilon, \eta) = N(\varepsilon, v)$ be so large that $n > N(\varepsilon, v)$ implies that $|s_n - s| < \xi(\varepsilon)$. Then, for $n > N(\varepsilon, v)$

$$|(s_n - a) - (s - a)| = |s_n - s| < \xi(\varepsilon, v)$$

(A.36)

for all $a \in [0, s]$. Thus, we have

$$|(s_n - a)^{1/2} - (s - a)^{1/2}| < \frac{\delta(\varepsilon)}{2\eta(\varepsilon)}$$

(A.37)

for all $n > N(\varepsilon)$ and $a \in [0, s]$. Thus, for all $|y| < \eta(\varepsilon)$, we have

$$|(t - 2(s_n - a)^{1/2}y) - (t - 2(s - a)^{1/2}y)| = 2|(s_n - a)^{1/2} - (s - a)^{1/2}| < \delta(\varepsilon, v)$$

(A.38)

for all $t \in \mathbb{R}$. Thus, for $n > N(\varepsilon, v)$, (A.36)-(A.38) imply that

$$\int_{|y| < \eta(\varepsilon)} |y|e^{-y^2} |v(a, t - 2(s_n - a)^{1/2}y) - v(a, t - 2(s - a)^{1/2}y)| dy < \varepsilon \int_{\mathbb{R}} |y|e^{-y^2} dy$$

(A.39)

uniformly in $a \in (0, s)$ and $t \in \mathbb{R}$. Combining (A.33), (A.34) and (A.39), we see that if $n > N(\varepsilon, v)$, then

$$\int_{\mathbb{R}} |y|e^{-y^2} |v(a, t - 2(s_n - a)^{1/2}y) - v(a, t - 2(s - a)^{1/2}y)| dy \leq \varepsilon + \varepsilon \int_{\mathbb{R}} |y|e^{-y^2} dy$$

(A.40)

for all $t \in \mathbb{R}, a \in (0, s)$, which proves (A.31). Case (b) when $s_n \leq s$ for all $n$ is handled similarly.

The next proposition is needed to show differentiability in $s$. □
Proposition A.1.12. Suppose that \( v \in X \cap C((0, T], \text{BUC}^1(\mathbb{R})) \) and there is a constant \( C \) such that
\[
\left\| \frac{\partial}{\partial t} v(s, \cdot) \right\|_\infty \leq C \left( \frac{1}{\sqrt{s}} + 1 \right).
\]
Then, for every \( t \in \mathbb{R} \),
\[
s \mapsto \int_0^s e^{(s-a)\Delta} v(a, t) da
\]
is differentiable at every \( s > 0 \), and
\[
\frac{\partial}{\partial s} \int_0^s e^{(s-a)\Delta} v(a, t) da = \int_0^s \frac{\partial}{\partial s} (e^{(s-a)\Delta} v(a, t)) da + v(s, t).
\]
Moreover, for every \( s > 0 \),
\[
t \mapsto \int_0^s e^{(s-a)\Delta} v(a, t) da
\]
is twice differentiable, and
\[
\frac{\partial^2}{\partial t^2} \int_0^s e^{(s-a)\Delta} v(a, t) da = \int_0^s \frac{\partial^2}{\partial t^2} e^{(s-a)\Delta} v(a, t) da.
\]
\[\text{Proof.}\] Fix \( t^* \in \mathbb{R} \) and fix an \( s > 0 \). We need to show that
\[
\frac{1}{h} \left( \int_0^{s+h} e^{(s+h-a)\Delta} v(a, t^*) da - \int_0^s e^{(s-a)\Delta} v(a, t^*) da \right)
\]
\[
\to \int_0^s \frac{\partial}{\partial s} (e^{(s-a)\Delta} v(a, t^*)) da + v(s, t^*) \tag{A.41}
\]
as \( h \to 0 \). We consider two cases: (a) \( h > 0 \) and (b) \( h < 0 \). In case (a), we have
\[
\frac{1}{h} \left( \int_0^{s+h} e^{(s+h-a)\Delta} v(a, t^*) da - \int_0^s e^{(s-a)\Delta} v(a, t^*) da \right)
\]
\[
= \int_0^s \frac{1}{h} \left( e^{(s+h-a)\Delta} v(a, t^*) - e^{(s-a)\Delta} v(a, t^*) \right) da
\]
\[
+ \frac{1}{h} \int_s^{s+h} e^{(s+h-a)\Delta} v(a, t^*) da \tag{A.42}
\]
\[
= A_h + B_h
\]
Now,
\[ B_h - v(s, t^*) = \frac{1}{h} \int_s^{s+h} (e^{(s+h-a)\Delta} v(a, t^*) - v(s, t^*)) \, da. \]

Hence
\[ |B_h - v(s, t^*)| \leq \frac{1}{h} \int_s^{s+h} |e^{(s+h-a)\Delta} v(a, t^*) - v(s, t^*)| \, da \]
\[ \leq \sup_{a \in [s,s+h]} \| e^{(s+h-a)\Delta} v(a, \cdot) - v(s, \cdot) \|_{\infty} \]

Thus, to show that \( |B_h - v(s, t^*)| \to 0 \) as \( h \to 0 \), it suffices to show that
\[ \sup_{a \in [s,s+h]} \| e^{(s+h-a)\Delta} v(a, \cdot) - v(s, \cdot) \|_{\infty} \to 0 \]  \hspace{1cm} (A.43)
as \( h \to 0 \). By definition of \( e^{s\Delta} \), we have
\[ e^{(s+h-a)\Delta} v(a, t) - v(s, t) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2} (v(a, t - 2(s + h - a)^{1/2}y) - v(s, t)) \, dy \]
\[ = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2} (v(a, t - 2(s + h - a)^{1/2}y) - v(a, t)) \, dy \]
\[ + (v(a, t) - v(s, t)) \] \hspace{1cm} (A.44)

Thus
\[ |e^{(s+h-a)\Delta} v(a, t) - v(s, t)| \leq \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2} |v(a, t - 2(s + h - a)^{1/2}y) - v(a, t)| \, dy \]
\[ + \| v(a, \cdot) - v(s, \cdot) \|_{\infty}. \] \hspace{1cm} (A.45)

Since \( \sup_{a \in [s,s+h]} \| v(a, \cdot) - v(s, \cdot) \|_{\infty} \to 0 \) as \( h \to 0 \), to show (A.43), it suffices by (A.45) to show that
\[ \sup_{a \in [s,s+h], t \in \mathbb{R}} \left( \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2} |v(a, t - 2(s + h - a)^{1/2}y) - v(a, t)| \, dy \right) \to 0 \] \hspace{1cm} (A.46)
as \( h \searrow 0 \). We have
\[
\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2} |v(a, t - 2(s + h - a)^{1/2}y) - v(a, t)| dy \\
\leq ||v||^* \frac{2}{\sqrt{\pi}} \int_{|y| \geq \eta} e^{-y^2} dy \\
+ \frac{1}{\sqrt{\pi}} \int_{|y| < \eta} e^{-y^2} |v(a, t - 2(s + h - a)^{1/2}y) - v(a, t)| dy
\]

(A.47)

Arguing as in the proof of Proposition A.1.11 (see (A.33) - (A.40)), for any \( \varepsilon > 0 \), we first choose \( \eta \) to make the first term in (A.47) smaller than \( \varepsilon \). Then, for all \( h \) sufficiently small (depending upon \( \eta \)), the second term in (A.47) will also be smaller than \( \varepsilon \) uniformly in \( a \) and \( t \).

It remains to show that
\[
A_h \to \int_0^s \frac{\partial}{\partial s} e^{(s-a)\Delta} v(a, t^*) da
\]

(A.48)
as \( h \searrow 0 \). We have
\[
A_h = \int_0^s \frac{1}{h} \left( e^{(s-h-a)\Delta} v(a, t^*) - e^{(s-a)\Delta} v(a, t^*) \right) da.
\]

(A.49)

For every \( a \in (0, s) \) and fixed \( t^* \in \mathbb{R} \), Lemma A.1.10 and the assumptions on \( v \) tell us that
\[
\frac{1}{h} \left( e^{(s+h-a)\Delta} v(a, t^*) - e^{(s-a)\Delta} v(a, t^*) \right) \to \frac{\partial}{\partial s} e^{(s-a)\Delta} v(a, t^*)
\]

(A.50)

Thus, if we can show that
\[
\left| \frac{1}{h} \left( e^{(s+h-a)\Delta} v(a, t^*) - e^{(s-a)\Delta} v(a, t^*) \right) \right| \leq g(a) \in L^1((0, s)),
\]

(A.51)

then (A.50), (A.51) and the dominated convergence theorem imply
\[
\int_0^s \frac{1}{h} \left( e^{(s+h-a)\Delta} v(a, t^*) - e^{(s-a)\Delta} v(a, t^*) \right) da \to \int_0^s \frac{\partial}{\partial s} \left( e^{(s-a)\Delta} v(a, t^*) \right) da.
\]

(A.52)
as \( h \searrow 0 \), and so (A.48) is true. To verify (A.51), note that
\[
\frac{1}{h} \left( e^{(s + h - a) \Delta v(a, t^*)} - e^{(s - a) \Delta v(a, t^*)} \right) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2} \left( \frac{1}{h} v(a, t^* - 2(s + h - a)^{1/2} y) - v(a, t^* - 2(s - a)^{1/2}) \right) dy.
\] (A.53)

Now, since \( v(a, t) \) is differentiable in \( t \), we have
\[
\frac{1}{h} \left( v(a, t^* - 2(s + h - a)^{1/2} y) - v(a, t^* - 2(s - a)^{1/2}) \right)\]
\[
= \frac{1}{h} \int_{(s-a)}^{(s+h-a)} \frac{d}{dx} v(a, t^* - 2x^{1/2}) dx
\]
\[
= \frac{1}{h} \int_{(s-a)}^{(s+h-a)} \left( \frac{\partial}{\partial t} v(a, t^* - 2x^{1/2}) \right) (-x^{-1/2}) dx
\]
\[
= -\frac{y}{h} \int_{(s-a)}^{(s+h-a)} \left( \frac{\partial}{\partial t} v(a, t^* - 2x^{1/2}) \right) \frac{1}{\sqrt{x}} dx.
\] (A.54)

Hence, using the assumption on the growth of \( \frac{\partial}{\partial t} v(a, t) \), we have
\[
\left| \frac{1}{h} \left( v(a, t^* - 2(s + h - a)^{1/2} y) - v(a, t^* - 2(s - a)^{1/2}) \right) \right|
\]
\[
\leq \frac{|y|}{h} \int_{(s-a)}^{(s+h-a)} \left| \left( \frac{\partial}{\partial t} v(a, t^* - 2x^{1/2}) \right) \right| \frac{1}{\sqrt{x}} dx
\]
\[
\leq \frac{|y|}{h} \int_{(s-a)}^{(s+h-a)} \left\| \frac{\partial}{\partial t} v(a, \cdot) \right\|_{\infty} \frac{1}{\sqrt{x}} dx
\]
\[
\leq C \frac{|y|}{h} \int_{(s-a)}^{(s+h-a)} \left( \frac{1}{\sqrt{a}} + 1 \right) \frac{1}{\sqrt{x}} dx
\]
\[
\leq C |y| \left( \frac{1}{\sqrt{a}} + 1 \right) \frac{1}{\sqrt{(s - a)}}.
\] (A.55)

Combining (A.53) and (A.55), we see that
\[
\left| \frac{1}{h} \left( e^{(s + h - a) \Delta v(a, t^*)} - e^{(s - a) \Delta v(a, t^*)} \right) \right|
\]
\[
\leq \frac{C}{\sqrt{\pi}} \left( \int_{\mathbb{R}} |y| e^{-y^2} dy \right)
\]
\[
\times \left( \frac{1}{\sqrt{a}} + 1 \right) \frac{1}{\sqrt{(s - a)}}.
\] (A.56)
Since the last term in (A.56) is in $L^1((0, s))$, we have (A.51). Case (b) when $h < 0$ is handled similarly.

For the differentiability in $t$, we follow the same pattern. Because of the assumption that $v(s, t) \in C((0, T], BUC^1(\mathbb{R}))$, we have for $a < s$

$$\frac{\partial}{\partial t} e^{(s-a)\Delta} v(a, t) = e^{(s-a)\Delta} v_t(a, t).$$

Thus, Proposition A.1.11 implies that

$$\frac{\partial}{\partial t} \int_0^s e^{(s-a)\Delta} v(a, t) \, da = \int_0^s \frac{\partial}{\partial t} e^{(s-a)\Delta} v(a, t) \, da = \int_0^s e^{(s-a)\Delta} v_t(a, t) \, da. \quad (A.57)$$

We want to show that

$$\frac{\partial}{\partial t} \int_0^s e^{(s-a)\Delta} v_t(a, t) \, da = \int_0^s \frac{\partial}{\partial t} e^{(s-a)\Delta} v_t(a, t) \, da. \quad (A.58)$$

We consider difference quotients in two cases: (a) $h > 0$ and (b) $h < 0$. In case (a), we have

$$\frac{1}{h} \int_0^s e^{(s-a)\Delta} (v_t(a, t + h) - v_t(a, t)) \, da = \int_0^s \frac{1}{h} \left( e^{(s-a)\Delta} v_t(a, t + h) - v_t(a, t) \right) \, da. \quad (A.59)$$

Now, for every $a \in (0, s)$, $t \mapsto e^{(s-a)\Delta} v_t(a, t)$ is differentiable, so

$$\frac{1}{h} \left( e^{(s-a)\Delta} v_t(a, t + h) - e^{(s-a)\Delta} v_t(a, t) \right) \rightarrow \left( \frac{\partial}{\partial t} e^{(s-a)\Delta} v_t(a, t) \right) \quad (A.60)$$

as $h \to 0$. If we can show

$$\left| \frac{1}{h} \left( e^{(s-a)\Delta} v_t(a, t + h) - e^{(s-a)\Delta} v_t(a, t) \right) \right| \leq g(a) \in L^1(0, s), \quad (A.61)$$

then the dominated convergence theorem applies, and so

$$\int_0^s \frac{1}{h} \left( e^{(s-a)\Delta} v_t(a, t + h) - e^{(s-a)\Delta} v_t(a, t) \right) \rightarrow \int_0^s \frac{\partial}{\partial t} e^{(s-a)\Delta} v_t(a, t) \, da \quad (A.62)$$
as \( h \to 0 \). To verify (A.61), we have

\[
\frac{1}{h} \left| e^{(s-a)\Delta} v_t(a, t + h) - e^{(s-a)\Delta} v_t(a, t) \right| = \frac{1}{h} \left| \int_t^{t+h} \frac{d}{dx} \left( e^{(s-a)\Delta} v_t(a, x) \right) dx \right|
\]

\[
\leq \frac{1}{h} \int_t^{t+h} \left| \frac{\partial}{\partial t} e^{(s-a)\Delta} v_t(a, x) \right| dx \quad (A.63)
\]

\[
\leq \frac{1}{h} \int_t^{t+h} \frac{C_1}{\sqrt{s-a}} \| v_t(a, \cdot) \|_\infty dx
\]

\[
\leq \frac{C_1}{\sqrt{s-a}} \left( 1 + \frac{1}{\sqrt{a}} \right) \in L^1((0, s)),
\]

which finishes the proof in case (a). The argument for case (b) is similar.

Next, we need some information about about \( V \). Let us recall the assumptions (DV1-7) on \( V \):

(DV1) \( V \in C^3(\mathbb{R} \times \mathbb{R}, \mathbb{R}) \)

(DV2) \( V \) is 1 periodic in \( t \): \( V(t + 1, q) = V(t, q) \)

(DV3) \( q = 0 \) and \( q = 1 \) are non-degenerate local maxima of \( V \) for every \( t \). Moreover, we assume \( V(t, 0) = 0 \) and there exists a constant \( c_1 > 0 \) such that \( V(t, 1) \geq c_1 \) for all \( t \).

(DV4) \( q = \frac{1}{2} \) is a non-degenerate local minimum of \( V \), and there is a constant \( c_2 < 0 \) such that for every \( t \), \( V(t, \frac{1}{2}) \leq c_2 \).

(DV5) \( V_q(t, x) \neq 0 \) for \( x \in (1/2, 1) \), and there is a constant \( \Lambda > 0 \) such that \( V(t, x) \leq -\Lambda x^2 \) for \( 0 \leq x \leq 1/2 \).

(DV6) There are constants \( \alpha_0, \beta > 0 \) such that \( V(t, q) < -\alpha_0 q^2 + \beta \).

(DV7) \( V_{qq}, V_{tq} \) and \( V_{qqq} \) are bounded.
Lemma A.1.13. Suppose that $V$ satisfied (DV1-7). Then, if $w \in X$, then $s \mapsto V_q(t, w(s, t)) \in X$, and there is a constant $K$ such that $\|V_q(\cdot, w(s, \cdot)) - V_q(\cdot, v(s, \cdot))\|_\infty \leq K\|w(s, \cdot) - v(s, \cdot)\|_\infty$.

**Proof.** Because of the regularity of $V$, we have

$$V_q(t, w(s, t)) - V_q(t, v(s, t)) = \int_0^1 \frac{d}{dx}\left(V_q(t, v(s, t) + x(w(s, t) - v(s, t))\right)dx$$

Thus, we have for all $t \in \mathbb{R}$

$$|V_q(t, w(s, t)) - V_q(t, v(s, t))| \leq \left(\int_0^1 \max_{t \in [0,1], q \in \mathbb{R}} |V_{qq}(t, q)| dx\right) |w(s, t) - v(s, t)|$$

$$\leq K\|w(s, \cdot) - v(s, \cdot)\|_\infty$$ (A.64)

Therefore, $\|V_q(\cdot, w(s, \cdot)) - V_q(\cdot, v(s, \cdot))\|_\infty \leq K\|w(s, \cdot) - v(s, \cdot)\|_\infty$. Moreover, if $s_n \to s$, repeating the argument above yields

$$\|V_q(\cdot, w(s_n, \cdot)) - V_q(\cdot, w(s, \cdot))\|_\infty \leq K\|w(s_n, \cdot) - w(s, \cdot)\|_\infty$$ (A.65)

which completes the proof. □

Corollary A.1.14. For any $w \in X$, if $v(s, t) := V_q(t, w(s, t))$, then $v \in X$, and $\|v\|^* \leq K\|w\|^*$

**Proof.** We simply take $v(s, t) \equiv 0$ above, making use of the fact that $V_q(t, 0) \equiv 0$. □

Let $T > 0$ be arbitrary. For any $u \in BUC(\mathbb{R})$, we define a map $\Phi_u : X \to X$ by

$$\Phi_u(w)(s, t) := e^{s\Delta}u(t) + \int_0^s e^{(s-a)\Delta}V_q(t, w(a, t))da$$ (A.66)
Letting \( v(s, t) := V_q(t, w(s, t)) \), we see that Lemmas (A.1.6) and (A.1.7) and Proposition (A.1.11) together imply that \( \Phi_u \) maps \( X \) to itself. Using (PDE), one can show that a solution of (PDE) will be a fixed point of \( \Phi_u \). Thus, we look for solutions of (PDE) as fixed points of \( \Phi_u \). (Compare [23], Chapter 15.)

**Lemma A.1.15.** For all \( T \) sufficiently small, \( \Phi_u \) is a contraction of \( X \).

**Proof.** By Lemmas A.1.6 and A.1.13, we have

\[
|\Phi_u(w)(s, t) - \Phi_u(v)(s, t)| = \left| \int_0^s e^{(s-a)\Delta} (V_q(t, w(s, t)) - V_q(t, v(s, t))) \, da \right|
\]

\[
\leq \int_0^T \left| e^{(s-a)\Delta} (V_q(t, w(a, t)) - V_q(t, v(a, t))) \right| \, da
\]

\[
\leq \int_0^T \| V_q(\cdot, w(a, \cdot)) - V_q(\cdot, v(a, \cdot)) \|_\infty \, da
\]

\[
\leq \int_0^T K \| w(a, \cdot) - v(a, \cdot) \|_\infty \, da
\]

\[
\leq KT \| w - v \|^* \tag{A.68}
\]

Taking \( T < 1/K \), we are finished. \( \square \)

Thus, for every \( u \in BUC(\mathbb{R}) \), there is a fixed point \( w_u \in X \) of \( \Phi_u \), i.e.

\[
w_u(s, t) = e^{s\Delta} u(t) + \int_0^s e^{(s-a)\Delta} V_q(t, w_u(a, t)) \, da. \tag{A.69}
\]

Moreover, since \( \| \Phi_u(w) - \Phi_u(v) \|^* \leq \lambda \| w - v \|^* \) for \( \lambda \in (0, 1) \) independent of \( u \), \( w_u \) is continuous in \( u \), i.e. for any \( \varepsilon > 0 \), there is a \( \delta > 0 \) such if \( \| u - u_1 \|_\infty < \delta \), then \( \| w_u - w_{u_1} \|^* < \varepsilon \). (See, for example, [5] or [11].)

Next, we show that \( w_u \) satisfies (PDE). First, we show that \( \| w_u(s, \cdot) - u(\cdot) \|_\infty \to 0 \) as \( s \searrow 0 \). By Lemma A.1.7 and (A.69), it suffices to show that

\[
\int_0^s e^{(s-a)\Delta} V_q(t, w(a, t)) \, da \to 0
\]
as \( s \searrow 0 \) uniformly in \( t \). We have

\[
\left| \int_0^s e^{(s-a)\Delta} V_q(t, w(a, t)) \, da \right| \leq \int_0^s \| V_q(t, w(a, t)) \|_\infty \, da \tag{A.70}
\]

\[
\leq sK\| w \|^*.
\]

Thus, the initial condition is satisfied.

Next, taking \( v(s, t) := V_q(t, w(s, t)) \), Lemma A.1.9 and Proposition A.1.11 imply that \( w \) given by (A.69) is differentiable in \( t \) for every \( s > 0 \). Moreover, \( \frac{\partial}{\partial t} w(s, t) \) is continuous in \( t \) and \( s \). Next, by Proposition A.1.11 we know that

\[
\left| \frac{\partial}{\partial t} w(s, t) \right| \leq \left| \frac{\partial}{\partial t} e^{s\Delta} u(t) \right| + \int_0^s \left| \frac{\partial}{\partial t} e^{(s-a)\Delta} v(a, t) \right| \, da
\]

\[
\leq \left\| \frac{\partial}{\partial t} e^{s\Delta} u \right\|_\infty + \int_0^s \left| \frac{\partial}{\partial t} e^{(s-a)\Delta} v(a, t) \right| \, da
\]

\[
\leq \frac{1}{\sqrt{s}}\| u \|_\infty + \int_0^s \frac{1}{\sqrt{s-a}}\| v(a, \cdot) \|_\infty \, da \tag{A.71}
\]

\[
\leq \frac{1}{\sqrt{s}}\| u \|_\infty + K\| w \|^* \int_0^s \frac{1}{\sqrt{s-a}} \, da
\]

\[
\leq \frac{1}{\sqrt{s}}\| u \|_\infty + K\| w \|^* T^{1/2}
\]

Since \( w(s, t) \) is differentiable in \( t \) for \( s > 0 \), we know that \( v(s, t) := V_q(t, w(s, t)) \) is differentiable, and

\[
\frac{\partial}{\partial t} v(s, t) = V_{tq}(t, w(s, t)) + V_{qq}(t, w(s, t)) \frac{\partial}{\partial t} w(s, t).
\]

But then we have

\[
\left\| \frac{\partial}{\partial t} v(s, \cdot) \right\|_\infty \leq K + K\left\| \frac{\partial}{\partial t} w(s, \cdot) \right\|_\infty
\]

\[
\leq K \left( 1 + \left( \frac{1}{\sqrt{s}}\| u \|_\infty + K\| w \|^* T^{1/2} \right) \right) \tag{A.72}
\]

\[
\leq C \left( 1 + \frac{1}{\sqrt{s}} \right) . \tag{A.73}
\]
Thus, \( v(s, t) := V_q(t, w(s, t)) \) satisfies the assumptions of Proposition A.1.12. Thus, \( w(s, t) \) is differentiable in \( s \) for \( s > 0 \), and

\[
\frac{\partial}{\partial s} w(s, t) = \frac{\partial}{\partial s} e^{s\Delta} u(t) + \int_0^s \frac{\partial}{\partial s} e^{(s-a)\Delta} v(a, t) da + v(s, t) \\
= \frac{\partial^2}{\partial t^2} e^{s\Delta} u(t) + \int_0^s \frac{\partial^2}{\partial t^2} e^{(s-a)\Delta} v(a, t) da + V_q(t, w(s, t)) \tag{A.74} \\
= \frac{\partial^2}{\partial t^2} (w(s, t) - \int_0^s e^{(s-a)\Delta} v(a, t) da) \\
+ \int_0^s \frac{\partial^2}{\partial t^2} (e^{(s-a)\Delta} v(a, t)) da + V_q(t, w(s, t)) \\
= \frac{\partial^2}{\partial t^2} w(s, t) + V_q(t, w(s, t)).
\]

Therefore, \( w \) satisfies (PDE). In addition, the solution depends continuously on the initial condition \( u \). We next turn to showing that \( w_s \) and \( w_{tt} \) are continuous in \( t \) and \( s \).

**Lemma A.1.16.** Suppose that \( v \in X \cap C([0, T], BUC^1(\mathbb{R})) \) and there is a constant \( C \) such that

\[
\left\| \frac{\partial}{\partial t} v(s, \cdot) \right\|_{\infty} \leq C \left( \frac{1}{\sqrt{s}} + 1 \right).
\]

Then for fixed \( t \), \( s \mapsto \int_0^s \frac{\partial}{\partial s} e^{(s-a)\Delta} v(a, t) da \) is continuous for \( s > 0 \). Similarly, for fixed \( s > 0 \), \( t \mapsto \int_0^s \frac{\partial}{\partial s} e^{(s-a)\Delta} v(a, t) da \) is continuous. Moreover, the function of \( s \) given by \( s \mapsto \int_0^s \frac{\partial}{\partial s} e^{(s-a)\Delta} v(a, t) da \) is bounded by a constant depending only on \( s \).

**Proof.** Let \( s_n \to s > 0 \). We consider two cases: (a) \( s_n \geq s \) and (b) \( s_n \leq s \) for all
n. We prove the lemma for case (a). The proof for (b) is similar. We have

\[
\int_0^{s_n} \frac{\partial}{\partial s} e^{(s_n-a)\Delta} v(a,t) da - \int_0^s \frac{\partial}{\partial s} e^{(s-a)\Delta} v(a,t) da
= \int_0^s \frac{\partial}{\partial s} \left( e^{(s_n-a)\Delta} v(a,t) da - e^{(s-a)\Delta} v(a,t) \right) da
+ \int_s^{s_n} \frac{\partial}{\partial s} e^{(s_n-a)\Delta} v(a,t) da. \tag{A.75}
\]

Now, using Lemmas A.1.9 and A.1.10, we see

\[
\left| \int_0^{s_n} \frac{\partial}{\partial s} e^{(s_n-a)\Delta} v(a,t) da \right| \leq \int_0^{s_n} \left\| \frac{\partial}{\partial s} e^{(s_n-a)\Delta} v(a,t) \right\|_\infty da \\
\leq \int_0^{s_n} \frac{1}{\sqrt{s_n-a}} \left\| \frac{\partial}{\partial t} v(a,\cdot) \right\|_\infty da \\
\leq \frac{C}{\sqrt{s_n-a}} \left( 1 + \frac{1}{\sqrt{a}} \right) \int_0^{s_n} \frac{1}{\sqrt{s_n-a}} da \\
\leq C \|v\|^* \left( 1 + \frac{1}{\sqrt{s}} \right) \leq \int_0^{s_n} \frac{1}{\sqrt{s_n-a}} da \\
= C(s)(s_n-s)^{1/2}
\]

which goes to 0 as \( n \to \infty \). Next, again using Lemmas A.1.9 and A.1.10, we have

\[
\left| \int_0^s \frac{\partial}{\partial s} \left( e^{(s_n-a)\Delta} v(a,t) da - e^{(s-a)\Delta} v(a,t) \right) da \right|
\leq \int_0^s \left\| \frac{\partial}{\partial s} \left( e^{(s_n-s)\Delta} v(a,t) - v(a,t) \right) \right\|_\infty da \tag{A.77}
\]

\[
\leq \int_0^s \frac{1}{\sqrt{s-a}} \left\| \frac{\partial}{\partial t} \left( e^{(s_n-s)\Delta} v(a,t) - v(a,t) \right) \right\|_\infty da \\
= \int_0^s \frac{1}{\sqrt{s-a}} \left\| e^{(s_n-s)\Delta} v_2(a,t) - v_2(a,t) \right\|_\infty da,
\]

where the \( v_2(a,t) \) denotes the derivative with respect to the second variable. For
0 < \delta < s/2 free for the moment, we have

\[
\int_0^s \frac{1}{\sqrt{s-a}} \left\| (e^{(s_n-a)\Delta} v_2(a,t) - v_2(a,t)) \right\|_\infty da \\
\leq \int_0^\delta \frac{2}{\sqrt{s-a}} \| v_2(a,\cdot) \|_\infty da \\
+ \int_\delta^s \frac{1}{\sqrt{s-a}} \| e^{(s_n-a)\Delta} v(a,\cdot) - v(a,\cdot) \|_\infty da \\
\leq \int_0^\delta \frac{C}{\sqrt{s-a}} \left( 1 + \frac{1}{\sqrt{a}} \right) da \\
+ \sup_{a \in [\delta,T]} \| e^{(s_n-a)\Delta} v_2(a,\cdot) - v_2(a,\cdot) \|_\infty \int_\delta^s \frac{1}{\sqrt{s-a}} da 
\tag{A.78}
\]

For any \varepsilon > 0, pick \delta = \delta(\varepsilon) < s/2 such that the first term is < \varepsilon. For this \delta(\varepsilon), note that \( v_2([\delta,T],\cdot) \subset BUC(\mathbb{R}) \) is compact. Thus, Lemma A.1.8 implies that for all \( n \) sufficiently large (depending on \( \delta(\varepsilon) \) and \( \varepsilon \)) the second term is < \varepsilon.

Combining (A.76) and (A.78), we see that for any \varepsilon > 0, we have for all sufficiently large \( n \) that

\[
\left| \int_0^{s_n} e^{(s_n-a)\Delta} v(a,t) da - \int_0^s e^{(s-a)\Delta} v(a,t) da \right| < \varepsilon.
\]

It remains to show the boundedness. We have

\[
\left| \int_0^s \frac{\partial}{\partial s} e^{(s-a)\Delta} v(a,t) da \right| \leq \int_0^s \frac{1}{\sqrt{s-a}} \left\| \frac{\partial}{\partial t} v(a,t) \right\|_\infty da 
\tag{A.79}
\]

\[
\leq C \int_0^s \frac{1}{\sqrt{s-a}} \left( 1 + \frac{1}{\sqrt{a}} \right) da.
\]
Since the last term in (A.79) is integrable, we see that as $s \downarrow 0$,
\[
\| \int_0^s \frac{\partial}{\partial s} e^{(s-a)\Delta} v(a,t) \|_\infty \rightarrow 0.
\]

Notice that because \( w(s,t) = e^{s\Delta} u(t) + \int_0^s e^{(s-a)\Delta} V_q(t, w(a,t)) da, \) for all \( s > 0, \)
\[
|w_s(s,t)| \leq \|u\|_\infty s + \| \int_0^s \frac{\partial}{\partial s} e^{(s-a)\Delta} V_q(\cdot, w(a,\cdot)) da \|_\infty + \|V_q(\cdot, w(s,\cdot))\|_\infty.
\]

### A.2 The Case of $W^{1,2}(\mathbb{R})$ and $E_k$

In this section, we show that if the initial condition $u$ is in either $E_\infty := W^{1,2}(\mathbb{R})$ or $E_k$, $k = 1, 2, \ldots$, then the solution $s \mapsto w(s,t)$ remains in $E_k$ for $k = 1, 2, \ldots, \infty$.

We start by proving some lemmas describing the behavior of $V_q(t, w(s,t))$ when $w \in C([0,T], E_k)$.

**Lemma A.2.1.** If $w \in C([0,T], E_k)$, then $V_q(t, w(s,t)) \in C([0,T], E_k)$, and there is a constant $C_3$ such that
\[
\|V_q(\cdot, w(s,\cdot))\|_{E_k} \leq C_3 \|w(s,\cdot)\|_{E_k}
\]

for $k = 1, 2, \ldots, \infty$.

**Proof.** Let $u_1, u_2 \in E_k$. Then
\[
|V_q(t, u_1(t)) - V_q(t, u_2(t))| = \left| \int_0^1 \frac{d}{dx} V_q(t, u_1(t) + x(u_1(t) - u_2(t))) dx \right|
\]
\[
\leq \left( \int_0^1 \left| V_{qq}(t, u_1(t) + x(u_1(t) - u_2(t))) \right| dx \right) |u_1(t) - u_2(t)|
\]
\[
\leq K |u_1(t) - u_2(t)|
\]
Thus, we will have $\|V_q(t, u_1(t)) - V_q(t, u_2(t))\|_{L^2(-k,k)} \leq K\|u_1 - u_2\|_{L^2(-k,k)}$. Next, we need to look at $\frac{\partial}{\partial t} V_q(t, u_1(t)) - \frac{\partial}{\partial t} V_q(t, u_2(t))$. We have

$$\frac{\partial}{\partial t} \left(V_q(t, u_1(t))\right) = V_{tq}(t, u_1(t)) + V_{qq}(t, u_1(t))\dot{u}_1(t),$$

where $\dot{u} = \frac{\partial}{\partial t} u$. Thus,

$$\frac{\partial}{\partial t} V_q(t, u_1(t)) - \frac{\partial}{\partial t} V_q(t, u_2(t)) = V_{tq}(t, u_1(t)) - V_{tq}(t, u_2(t))$$

$$+ V_{qq}(t, u_1(t))\dot{u}_1(t) - V_{qq}(t, u_2(t))\dot{u}_2(t).$$

Since $V_{tqq}$ is bounded, we deal with the first two terms as in (A.80) to get

$$|V_{tq}(t, u_1(t)) - V_{tq}(t, u_2(t))| \leq C|u_1(t) - u_2(t)|$$  \hspace{1cm} (A.81)

Next, we have

$$|V_{qq}(t, u_1(t))\dot{u}_1(t) - V_{qq}(t, u_2(t))\dot{u}_2(t)| \leq |V_{qq}(t, u_1(t))||\dot{u}_1(t) - \dot{u}_2(t)|$$  \hspace{1cm} (A.82)

$$+ |V_{qq}(t, u_1(t)) - V_{qq}(t, u_2(t))| |\dot{u}_2(t)|.$$ 

Since $V_{qqq}$ is bounded, we deal with the last term in (A.82) as in (A.80) to get

$$|V_{qq}(t, u_1(t)) - V_{qq}(t, u_2(t))| \leq C|u_1(t) - u_2(t)|.$$  \hspace{1cm} (A.83)

Combining (A.81), (A.82), and (A.83) and using (DV7), we have

$$\left| \frac{\partial}{\partial t} \left(V_q(t, u_1(t))\right) - \frac{\partial}{\partial t} \left(V_q(t, u_2(t))\right) \right| \leq C\left(|u_1(t) - u_2(t)| + |u_1(t) - u_2(t)||\dot{u}_2(t)|\right.$$  

$$+ |\dot{u}_1(t) - \dot{u}_2(t)|.$$  \hspace{1cm} (A.84)
Thus,
\[
\int_{-k}^{k} \left( \frac{\partial}{\partial t} V_q(t, u_1(t)) - \frac{\partial}{\partial t} V_q(t, u_2(t)) \right) dt \leq C \left( \int_{-k}^{k} \left( 1 + |\dot{u}_2(t)|^2 \right) |u_1(t) - u_2(t)|^2 + |\dot{u}_1(t) - \dot{u}_2(t)|^2 \right) dt.
\]
(A.85)

Combining (A.80) and (A.85), we see that
\[
\|V_q(t, u_1(t)) - V_q(t, u_2(t))\|^2_{E_k} \leq C \left( \int_{-k}^{k} \left( 1 + |\dot{u}_2(t)|^2 \right) |u_1(t) - u_2(t)|^2 + |\dot{u}_1(t) - \dot{u}_2(t)|^2 \right) dt
\]
\[
\leq C \left( \|u_1 - u_2\|^2_{E_k} + \int_{-k}^{k} |\dot{u}_2(t)|^2 |u_1(t) - u_2(t)|^2 dt \right)
\]
\[
\leq C \left( \|u_1 - u_2\|^2_{E_k} + \|u_1 - u_2\|^2_{L^\infty(-k,k)} \int_{-k}^{k} |\dot{u}_2(t)|^2 dt \right)
\]
\[
\leq C \|u_1 - u_2\|^2_{E_k} \left( 1 + \int_{-k}^{k} |\dot{u}_2(t)|^2 dt \right).
\]
(A.86)

Thus, if \(w \in C([0, T], E_k)\), then (A.86) implies that \(v(s, t) := V_q(t, w(s, t)) \in C([0, T], E_k)\). Moreover, taking \(u_2 \equiv 0\), we see that there is a constant \(C_3\) such that
\[
\|v(s, \cdot)\|_{E_k} \leq C_3 \|w(s, \cdot)\|_{E_k}.
\]
(A.87)

Lemma A.2.2. \(\|e^{s\Delta} u - u\|_{E_k} \to 0\) as \(s \searrow 0\).

Proof. We consider first the case of \(k = \infty\). In this case, we use the Fourier transform. Notice that \(e^{s\Delta}\) is a multiplier operator, and
\[
\hat{e^{s\Delta} u}(\xi) = e^{-4\pi^2 s^2} \hat{u}(\xi).
\]
(A.88)
We see that
\[
\|e^{s\Delta} u - u\|_{W^{1,2}(\mathbb{R})}^2 = \|\hat{e}^{s\Delta} u - \hat{u}\|_{W^{1,2}(\mathbb{R})}^2
\]
\[
= \int_{\mathbb{R}} (1 + \xi^2) \left| \hat{e}^{s\Delta} u(\xi) - \hat{u}(\xi) \right|^2 d\xi
\]
\[
= \int_{\mathbb{R}} (1 + \xi^2) \left| e^{-4s^2\xi^2} - 1 \right|^2 \left| \hat{u}(\xi) \right|^2 d\xi.
\] (A.89)

Now, for every \( \xi \), as \( s \searrow 0 \), the integrand in the last integral in (A.89) goes to 0.

Since \( (1 + \xi^2)|\hat{u}(\xi)|^2 \in L^1(\mathbb{R}) \), the dominated convergence theorem implies that
\[
\int_{\mathbb{R}} (1 + \xi^2)|e^{-4s^2\xi^2} - 1|^2 |\hat{u}(\xi)|^2 d\xi \to 0
\]
as \( s \searrow 0 \), which implies (A.89).

For \( k < \infty \), notice that
\[
u(t) = \sum_{n=0}^{\infty} \left( a_n \sin \frac{n\pi t}{k} + b_n \cos \frac{n\pi t}{k} \right)
\] (A.90)
for \( u \in E_k \), and \( \|u\|_{E_k}^2 = \sum (1 + n^2)(a_n^2 + b_n^2) \). Moreover,
\[
e^{s\Delta} u(t) := \sum_{n=0}^{\infty} e^{(-\frac{n\pi}{k})^2 s} \left( a_n \sin \frac{n\pi t}{k} + b_n \cos \frac{n\pi t}{k} \right).
\] (A.91)
This can be seen by noting that both the formula given in Definition A.1.5 and (A.91) solve the heat equation with initial data \( u \). We need to show that \( \|e^{s\Delta} u - u\|_{E_k} \to 0 \) as \( s \searrow 0 \). This is equivalent to showing that
\[
\sum_{n=0}^{\infty} \left| 1 - e^{(-\frac{n\pi}{k})^2 s} \right|^2 (1 + n^2)(a_n^2 + b_n^2) \to 0.
\] (A.92)
Let \( \varepsilon > 0 \) be given. Then, since \( \sum_{n=0}^{\infty} (1 + n^2)(a_n^2 + b_n^2) < \infty \), there is an \( N(\varepsilon) \) such that
\[
\sum_{n=N(\varepsilon)}^{\infty} (1 + n^2)(a_n^2 + b_n^2) < \varepsilon.
\] (A.93)
Now, for each \( n < N(\varepsilon) \), there is an \( s_n(\varepsilon) \) such that if \( s < s_n(\varepsilon) \), then

\[
\left| 1 - e^{(-\frac{n\pi}{\varepsilon})^2 s} \right|^2 (1 + n^2)(a_n^2 + b_n^2) < \frac{\varepsilon}{N(\varepsilon)}.
\]  

(A.94)

Thus, for \( s < \min\{s_0(\varepsilon), s_1(\varepsilon), \ldots, s_{N(\varepsilon)}\} \), (A.93) and (A.94) imply

\[
\sum_{n=0}^{\infty} \left| 1 - e^{(-\frac{n\pi}{\varepsilon})^2 s} \right|^2 (1 + n^2)(a_n^2 + b_n^2) \leq \sum_{n=0}^{n=N(\varepsilon)} \frac{\varepsilon}{N(\varepsilon)} + \sum_{n=N(\varepsilon)+1}^{\infty} (1 + n^2)(a_n^2 + b_n^2) 
\leq \varepsilon + \varepsilon,
\]

which completes the proof.

We also have (as in the BUC(\( \mathbb{R} \)) case) the following corollary:

**Corollary A.2.3.** Let \( K \subset E_k \) be compact. Then

\[
\sup_{u \in K} \| e^{s\Delta} u - u \|_{E_k} \to 0
\]

as \( s \downarrow 0 \).

The next lemma is used to show that a mapping like \( \Phi_u \) maps \( C([0, T], E_k) \) into itself.

**Lemma A.2.4.** For every \( v \in C([0, T], E_k) \),

\[
s \mapsto \int_0^s e^{(s-t)\Delta} v(a, t) \, da \in C([0, T], E_k).
\]

**Proof.** First, note that \( \| e^{s\Delta} u \|_{E_k} \leq \| u \|_{E_k} \). Suppose that \( s_n \to s \) as \( n \to \infty \).
Suppose first that $s_n \geq s$ for all $n$. Then

\[
\left\| \int_0^{s_n} e^{(s_n-a)\Delta} v(a, \cdot) da - \int_0^s e^{(s-a)\Delta} v(a, \cdot) da \right\|_{E_k} \\
\leq \int_0^{s_n} \left\| e^{(s_n-a)\Delta} (e^{(s_n-s)\Delta} v(a, \cdot) - v(a, \cdot)) \right\|_{E_k} da \\
+ \int_{s_n}^s \left\| e^{(s_n-a)\Delta} v(a, \cdot) \right\|_{E_k} da \\
\leq \int_0^s \left\| e^{(s_n-s)\Delta} v(a, t) - v(a, t) \right\|_{E_k} da + \int_{s_n}^s \left\| v(a, \cdot) \right\|_{E_k} da \\
\leq T \sup_{a \in [0, s]} \left\| e^{(s_n-s)\Delta} v(a, \cdot) - v(a, \cdot) \right\|_{E_k} \\
+ (s_n - s) \max_{a \in [0, T]} \left\| v(a, \cdot) \right\|_{E_k}.
\]

(A.95)

Then, Corollary A.2.3 finishes the proof. \(\square\)

Now, we are ready to find fixed points of $\Phi_u$ where

\[
\Phi_u(w)(s, t) = e^{s\Delta} u(t) + \int_0^s e^{(s-a)\Delta} V_q(t, w(a, t)) da.
\]

(A.96)

The previous lemma assures that $\Phi_u$ maps $C([0, T], E_k)$ to itself. As a norm on $C([0, T], E_k)$, we take

\[
\|w\|^*_k := \sup_{s \in [0, T]} \|w(s, \cdot)\|_{E_k}.
\]

The difficulty here is that $\Phi_u$ isn’t globally Lipschitz as a map from $C([0, T], E_k)$ to itself. For any $R > 0$, we define

\[
\mathcal{A}_{R,T} := \{ w \in C([0, T], E_k) \mid \|w\|^*_k \leq R + 1 \}.
\]

(A.97)

**Lemma A.2.5.** There is a $T_R$ such that $\Phi_u : \mathcal{A}_{R,T_R} \to \mathcal{A}_{R,T_R}$ for all $u \in E_k$ with $\|u\|_{E_k} \leq R$. 
Proof. Let \( w \in \mathcal{A}_{R,T} \). Then,

\[
\|\Phi_u(w)(s, \cdot) - u\|_{E_k} \leq \|e^{s\Delta}u - u\|_{E_k} + \left\| \int_0^s e^{(s-a)\Delta}V_q(t, w(a, t))da \right\|_{E_k}.
\]

(A.98)

Now, \( \|e^{s\Delta}u - u\|_{E_k} \leq \|u\|_{E_k} \leq R \), and

\[
\left\| \int_0^s e^{(s-a)\Delta}V_q(t, w(a, t))da \right\|_{E_k} \leq \int_0^s \|V_q(t, w(a, t))\|_{E_k} da 
\]

(A.99)

\[
\leq C_3T(R + 1)
\]

since \( w \in \mathcal{A}_{R,T} \). Thus, taking \( T_R \) such that \( R + C_3T_R(R + 1) \leq R + 1 \), we have the lemma.

\( \Box \)

Next, we show that by making \( T_R \) smaller (if necessary), \( \Phi_u : \mathcal{A}_{R,T_R} \to \mathcal{A}_{R,T_R} \) is a contraction whenever \( \|u\|_{E_k} \leq R \).

Lemma A.2.6. There is a \( \tilde{T}_R \leq T_R \) such that \( \Phi_u : \mathcal{A}_{R,\tilde{T}_R} \to \mathcal{A}_{R,\tilde{T}_R} \) is a contraction whenever \( \|u\|_{E_k} \leq R \).

Proof. Suppose that \( \tilde{T}_R \leq T_R \), \( w_1, w_2 \in \mathcal{A}_{R,\tilde{T}_R} \), and \( \|u\|_{E_k} \leq R \). Then

\[
\|\Phi_u(w_1)(s, \cdot) - \Phi_u(w_2)(s, \cdot)\|_{E_k} \leq \int_0^s \|V_q(\cdot, w_1(a, \cdot) - V_q(\cdot, w_2(a, \cdot))\|_{E_k} da
\]

(A.100)

\[
\leq \tilde{T}_R \sup_{s \in [0, \tilde{T}_R]} \|V_q(\cdot, w_1(s, \cdot) - V_q(\cdot, w_2(s, \cdot))\|_{E_k}.
\]

Now, using (A.86), we know that

\[
\|V_q(\cdot, w_1(s, \cdot) - V_q(\cdot, w_2(s, \cdot))\|_{E_k}^2 \leq \|w_1(s, \cdot) - w_2(s, \cdot)\|_{E_k}^2 \left( 1 + \int_{-k}^k \|w_1(s, \cdot)\|_{E_k}^2 dt \right)
\]

(A.101)

\[
\leq C(R + 1)^2 \|w_1(s, \cdot) - w_2(s, \cdot)\|_{E_k}^2
\]
Combining (A.100) and (A.101), we see that there is a constant \( C \) independent of \( R \) such that
\[
\|\Phi_u(w_1)(s, \cdot) - \Phi_u(w_2)(s, \cdot)\|_{E_k} \leq C\tilde{T}_R(R + 1) \sup_{s \in [0, \tilde{T}_R]} \|w_1(s, \cdot) - w_2(s, \cdot)\|_{E_k}.
\]
Taking \( \tilde{T}_R < \min\{T_R, 1/((R + 1)C)\} \) gives us an appropriate \( \tilde{T}_R \).

Thus, for any \( u \in E_k \) with \( \|u\|_{E_k} \leq R \), there is a unique fixed point \( w_u \) of \( \Phi_u \) in \( A_{R, \tilde{T}_R} \). Thus, there is a solution of (PDE) with initial condition \( u \), at least for some interval \([0, s_u)\). Moreover, since the contraction is uniform for \( u \in E_k \) with \( \|u\|_{E_k} \), the fixed point depends continuously on \( u \): for any \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that if \( \|u - u_1\|_{E_k} \leq \delta \), then \( \|w_u - w_{u_1}\|_k^* < \varepsilon \). Notice, this implies (1) and (3) of Theorem A.1.1, at least on short time intervals. Once we know that a solution exists for all \( s \geq 0 \), we will have (1) and (3).

Next, we show that \( s \mapsto I_k(w_u(s, \cdot)) \) is decreasing.

**Lemma A.2.7.** For every \( s > 0 \),
\[
\frac{d}{ds}(I_k(w_u(s, \cdot))) = -\int_{-k}^{k} \left( \frac{\partial}{\partial s} w_u(s, t) \right)^2 dt.
\]

**Proof.** Let \( g(s) := I_k(w_u(s, \cdot)) \). It suffices to show that
\[
g'(s) = \int_{-k}^{k} \left( -w_{tt}w_s - V_q(t, w)w_s \right) dt.
\]

Let \( s^* > 0 \). Then
\[
\frac{1}{h} (g(s^* + h) - g(s^*)) = \int_{-k}^{k} \left( \frac{1}{2h} \left( w_t^2(s^* + h, t) - w_t^2(s^*, t) \right) \right) dt
\]
\[
- \int_{-k}^{k} \left( \frac{1}{h} \left( V(t, w(s^* + h, t)) - V(t, w(s^*, t)) \right) \right) dt
\]
\[
= A_h + B_h.
\]
Now, after integrating by parts (since at each \( s > 0, w \) is \( C^2 \) as a function of \( t \)), we have

\[
\int_{-k}^{k} w_t^2(s, t) dt = - \int_{-k}^{k} w(s, t)w_{tt}(s, t) dt. \tag{A.104}
\]

Thus, by (A.103) and (A.104), we know that

\[
A_h = \int_{-k}^{k} \frac{1}{2h} \left( w(s^* + h, t)w_{tt}(s^* + h, t) - w(s^*, t)w_{tt}(s^*, t) \right) dt
\]

\[
= \int_{-k}^{k} \frac{1}{2h} \left( (w(s^* + h, t) - w(s^*, t))w_{tt}(s^* + h, t)
\right.

\[
+ \left( w(s^* + h, t) - w(s^*, t) \right)w_{tt}(s^*, t) \right) dt \tag{A.105}
\]

\[
+ \int_{-k}^{k} \frac{1}{2h} \left( w(s^*, t)w_{tt}(s^* + h, t) - w(s^* + h, t)w_{tt}(s^*, t) \right) dt.
\]

Integration by parts shows us that the last integral (A.105) is 0. Thus,

\[
A_h = \int_{-k}^{k} \frac{1}{2h} \left( (w(s^* + h, t) - w(s^*, t))w_{tt}(s^* + h, t)
\right.

\[
+ \left( w(s^* + h, t) - w(s^*, t) \right)w_{tt}(s^*, t) \right) dt
\]

\[
= \int_{-k}^{k} \frac{1}{h} \left( w(s^* + h, t) - w(s^*, t) \right)w_{tt}(s^*, t) dt \tag{A.106}
\]

\[
+ \int_{-k}^{k} \frac{1}{2h} \left( w(s^* + h, t) - w(s^*, t) \right) \left( w_{tt}(s^* + h, t) - w_{tt}(s^*, t) \right) dt.
\]

Notice that for every \( t \), we have

\[
\frac{1}{h} \left( w(s^* + h, t) - w(s^*, t) \right) \rightarrow w_s(s^*, t) \quad \text{and} \quad \tag{A.107}
\]

\[
w_{tt}(s^* + h, t) - w_{tt}(s^*, t) \rightarrow 0
\]

as \( h \rightarrow 0 \). Next, \( w_{tt}(s, t) \) and \( w_s(s, t) \) are bounded by a constant depending only on \( s > 0 \). Therefore, the dominated convergence theorem, (A.106) and (A.107)
imply that
\[ A_h \to \int_{-k}^{k} -w_s(s^*, t)w_t(s^*, t)dt \quad (A.108) \]
as \( h \to 0 \). Next, we turn our attention to \( B_h \). Because \( w(s, t) \) is differentiable at every \( s > 0 \) for every \( t \),
\[ \frac{1}{h} \left( V(t, w(s^* + h, t)) - V(t, w(s^*, t)) \right) \to V_q(t, w(s^*, t))w_s(s^*, t) \quad (A.109) \]
as \( h \to 0 \). Moreover, since \( \|w_s(s, \cdot)\|_\infty < \infty \) for all \( s \) in a neighborhood of \( s^* \), we may again use the dominated convergence theorem to conclude that
\[ B_h = -\int_{-k}^{k} \left( \frac{1}{h} \left( V(t, w(s^* + h, t)) - V(t, w(s^*, t)) \right) \right) dt \]
\[ \to -\int_{-k}^{k} V_q(t, w(s^*, t))w_s(s^*, t)dt. \]

To see (2) of Theorem A.1.1, we set \( \varphi_s(u)(t) = w_u(s, t) \).

Lemma A.2.8. Let \( s_1, s_2 \geq 0 \) be sufficiently small that \( \varphi_{s_2}(\varphi_{s_1}(u)) \) and \( \varphi_{s_1+s_2}(u) \) exist. Then \( \varphi_{s_2}(\varphi_{s_1}(u))(t) = \varphi_{s_1+s_2}(u)(t) \).

Proof. Let \( w_1(s, t) = \varphi_s(\varphi_{s_1}(u))(t) \) and \( w_2(s, t) = \varphi_{s+s_1}(u)(t) \). Then \( w_1(0, t) = \varphi_{s_1}(t) = w_2(0, t) \). Since both \( w_1, w_2 \) solve the same partial differential equation, uniqueness implies that \( w_1(s, t) = w_2(s, t) \).

Using the semi-group property, if \( [0, T_u) \) is the maximal interval of existence of a solution, then we must have (as in the case of ordinary differential equations) \( \|\varphi_s(u)\|_{E_k} \not\to \infty \) as \( s \not\to T_u \). However, note that by Lemma A.2.7, \( I_k(\varphi_s(u)) \) is
decreasing, hence

\[ I_k(u) \geq I_k(\varphi_s(u)) = \int_{-k}^{k} \left( \frac{1}{2} \dot{\varphi}_s^2(u)(t) - V(t, \varphi_s(u)(t)) \right) dt \]

\[ \geq \int_{-k}^{k} \left( \frac{1}{2} \dot{\varphi}_s^2(u)(t) + \alpha_0 \varphi_s^2(u)(t) - \beta \right) dt \]

\[ \geq \min\{1/2, \alpha_0\} \|\varphi_s(u)\|_{E_k}^2 - 2\beta k. \]  

(A.110)

Thus, \( \|\varphi_s(u)\|_{E_k} \) is bounded, and so a solution exists for all \( s > 0 \). This implies that we have (1) - (3) of Theorem A.1.1. It remains to show that (4) of Theorem A.1.1 is satisfied. Throughout, \( u \in E_k \) will be fixed. For any \( a < b \), we define

\[ K^b_a := \{ u \in E_k \mid I'_k(u) = 0, \ I_k(u) \in [a, b] \} \]  

(A.111)

\[ K^b := \{ u \in E_k \mid I'_k(u) = 0, \ I_k(u) \leq b \} \]  

(A.112)

Since \( I_k \) satisfies the \((PS)\) condition, \( K^a_b \) is compact. Similarly, since \( I_k \) is bounded from below, \( K^b \) is compact. Let \( s_i \) be a sequence with \( s_i \to \infty \) as \( i \to \infty \). We wish to show that there is a subsequence \( s_{i_j} \to \infty \) and a solution \( \bar{u} \in E_k \) of \((HS)\) such that \( \|\varphi_{s_{i_j}}(u) - \bar{u}\|_{E_k} \to 0 \) as \( j \to \infty \). Let \( g(s) := I_k(\varphi_s(u)) \). We have the following important lemma:

**Lemma A.2.9.** For \( s > 0 \), we have

\[ \|I'_k(\varphi_s(u))\|_{E'_k} \leq \|\frac{\partial}{\partial s}(\varphi_s(u))\|_{L^2(-k,k)}. \]
Proof. Let \( \psi \in E_k \) and suppose that \( \|\psi\|_{E_k} \leq 1 \). Then, for \( s > 0 \)

\[
\begin{align*}
|I'_{k}(\varphi_{s}(u))\psi| &= \left| \int_{-k}^{k} \left( \varphi_{s}(u)\dot{\psi} - V_{q}(t, \varphi_{s}(u))\psi \right) dt \right| \\
&= \left| \int_{-k}^{k} \left( \ddot{\varphi}_{s}(u) + V_{q}(t, \varphi_{s}(u)) \right) \psi dt \right| \\
&\leq \int_{-k}^{k} \left| \frac{\partial}{\partial s} \varphi_{s}(u) \right| |\psi| dt \\
&\leq \left\| \frac{\partial}{\partial s} (\varphi_{s}(u)) \right\|_{L^{2}(-k,k)} \|\psi\|_{L^{2}(-k,k)} \\
&\leq \left\| \frac{\partial}{\partial s} (\varphi_{s}(u)) \right\|_{L^{2}(-k,k)}.
\end{align*}
\] (A.113)

\[\square\]

Now, since \( I_{k} \) is bounded from below and \( g(s) \) is decreasing, there is a constant \( c \) such that \( g(s) \to c \) as \( s \to \infty \). But then

\[
c - I_{k}(u) = \int_{0}^{\infty} g'(s)ds
\] (A.114)

Notice that this means there is a subsequence \( \tilde{s}_{i} \to \infty \) as \( i \to \infty \) such that \( g'(\tilde{s}_{i}) \to 0 \) as \( i \to \infty \). But then, by Lemmas A.2.7 and A.2.9, we have

\[
\|I'_{k}(\varphi_{\tilde{s}_{i}}(u))\|_{E_{k}'}^{2} \leq -g'(\tilde{s}_{i}),
\] (A.115)

hence \( \|I'_{k}(\varphi_{\tilde{s}_{i}}(u))\| \to 0 \) as \( i \to \infty \). Because \( I_{k} \) satisfies the (PS) condition, there is a subsequence \( \tilde{s}_{i_{j}} \) and a solution \( \tilde{u} \in E_{k} \) with \( \|\varphi_{\tilde{s}_{i_{j}}}(u) - \tilde{u}\|_{E_{k}} \to 0 \) as \( j \to \infty \). Thus, \( K_{c}^{-I_{k}(u)} \neq \emptyset \).

If there is a subsequence \( s_{i_{j}} \) with \( \|I'_{k}(\varphi_{s_{i_{j}}}(u))\| \to 0 \), then since \( I_{k} \) satisfies the (PS) condition, there is a further subsequence and a solution \( \bar{u} \in E_{k} \) of (HS) such that \( \|\varphi_{s_{i_{j}}}(u) - \bar{u}\| \to 0 \) as \( j \to \infty \), and so (4) of Theorem A.1.1 is satisfied. So, suppose that there is a \( \delta > 0 \) such that \( \|I'_{k}(\varphi_{s_{i}}(u))\| \geq \delta \). We wish to get a
contradiction. Notice that by (A.115) this means that \( g'(s_i) \leq -\delta^2 \). We will need the following lemma:

**Lemma A.2.10.** There is an \( \eta > 0 \) such that \( \| \varphi_{s_i}(u) - K^I_k(u) \| \geq \eta \).

**Proof.** If not, then there is a sequence \( u_i \in K^{I_k(u)} \) with \( \| \varphi_{s_i}(u) - u_i \|_{E_k} \to 0 \) as \( i \to \infty \). Since \( K^{I_k(u)} \) is compact, there is a subsequence \( u_{i_j} \in K^{I_k(u)} \) that converges to some \( \bar{u} \in K^{I_k(u)} \). But then

\[
\| \varphi_{s_{ij}}(u) - u \|_{E_k} \leq \| \varphi_{s_{ij}}(u) - u_{ij} \|_{E_k} + \| u_{ij} - \bar{u} \|_{E_k},
\]

so \( \varphi_{s_{ij}}(u) \to u \in K^{I_k(u)} \) as \( j \to \infty \). But then \( \| I'_k(\varphi_{s_{ij}})(u) \| \to \| I'_k(\bar{u}) \| = 0 \), which contradicts our assumption about the sequence \( \varphi_{s_i}(u) \). \( \square \)

Now, if there were intervals \( J_i \) containing \( s_i \) with lengths bounded away from zero and for which \( g'(s) \leq -\delta^2/2 \) whenever \( s \in J_i \), then \( g' \not\in L^1([0, \infty)) \). Therefore, there must be a sequence \( \tilde{s}_i \) such that \( \tilde{s}_i > s_i, \tilde{s}_i - s_i \to 0 \) as \( i \to \infty \) and \( g'(--) \to 0 \).

By Lemmas A.2.9 and A.2.7, \( \| I'_k(\varphi_{\tilde{s}_i}(u)) \| \to 0 \) as \( i \to \infty \). Therefore, there is a solution \( \bar{u} \in E_k \) of (HS) such that \( \varphi_{\tilde{s}_i}(u) \to \bar{u} \) in \( E_k \) as \( i \to \infty \). Thus, \( \| \varphi_{\tilde{s}_i}(u) - K^{I_k(u)} \|_{E_k} \to 0 \) as \( i \to \infty \). If we can show that \( \| \varphi_{\tilde{s}_i}(u) - \varphi_{s_i}(u) \|_{E_k} \to 0 \), we will have a contradiction to Lemma A.2.10. Notice that \( \varphi_s(u)(t) = e^{s\Delta}u(t) + \int_0^s e^{(s-a)\Delta}V_q(t, \varphi_a(u)(t))da \). Therefore, using Lemma A.2.1 and (A.110)

\[
\| \varphi_{\tilde{s}_i}(u) - \varphi_{s_i}(u) \|_{E_k} \leq \| e^{\tilde{s}_i\Delta}u - e^{s_i\Delta}u \|_{E_k} + \left\| \int_{s_i}^{\tilde{s}_i} e^{(s-a)\Delta}V_q(t, \varphi_a(u)(t))da \right\|_{E_k} \\
\leq \| e^{(\tilde{s}_i-s_i)\Delta}u - u \|_{E_k} + \int_{s_i}^{\tilde{s}_i} \| V_q(t, \varphi_a(u)(t)) \|_{E_k} da \\
\leq \| e^{(\tilde{s}_i-s_i)\Delta}u - u \|_{E_k} + \int_{s_i}^{\tilde{s}_i} C_3 \| \varphi_a(u) \|_{E_k} da \\
\leq \| e^{(\tilde{s}_i-s_i)\Delta}u - u \|_{E_k} + C_3(\tilde{s}_i - s_i)(I_k(u) + 2\beta k).
\]
But the last term above goes to 0 as $i \to \infty$, since $\bar{s}_i - s_i \to 0$ as $i \to \infty$. 
Bibliography


