1 Preliminaries

In this section, we investigate the existence of heteroclinic and homoclinic solutions of the Hamiltonian system

\[ (HS) \quad \ddot{q}(t) = -V_q(t, q(t)), \]

where \( V \) is a potential with finitely many wells, and \( q(t) \in \mathbb{R}^d \). Throughout, \( \| \cdot \|_{L^p} \) and \( \| \cdot \|_{W^{1,2}} \) will always be the standard norms on \( L^p(\mathbb{R}), W^{1,2}(\mathbb{R}) : = E \) respectively, while \( \| \cdot \|_{L^p(A)}, \| \cdot \|_{W^{1,2}(A)} \) will be the standard norms on \( L^p(A), W^{1,2}(A) \) for \( A \subset \mathbb{R} \). Next, \( \langle \cdot, \cdot \rangle_E \) will be the inner product in \( E \). Finally, \( B_r(y) \) will denote the ball of radius \( r \) centered at \( y \). From context, it will be clear what space \( B_r(y) \) is in. For example, if \( u \in E \), \( B_r(u) \subset E \), while if \( \xi \in \mathbb{R}^d \), then \( B_r(\xi) \subset \mathbb{R}^d \).

We look for solutions of (HS) as critical points of the functional

\[ I(q) := \int_{\mathbb{R}} \left( \frac{1}{2} |\dot{q}(t)|^2 - V(t, q(t)) \right) dt \]

for \( q \in \dot{E} : = W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^d) \). We make \( \dot{E} \) into a Hilbert space by using the inner product

\[ \langle u, v \rangle_E := \langle u(0), v(0) \rangle + \int_{\mathbb{R}} \langle \dot{u}(t), \dot{v}(t) \rangle dt, \]

where \( \langle \cdot, \cdot \rangle \) is the standard inner product in \( \mathbb{R}^d \). Moreover, we make the following assumptions on the potential \( V \):

(V1) \( V \in C^2(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}) \), and \( V \) is 1-periodic in \( t \).

(V2) There is a finite set of points \( K(V) = \{ \xi_1, \xi_2, \xi_3, \ldots \xi_k \} \) such that \( V(t, \xi_i) = 0 > V(t, x) \) for \( x \notin K(V) \) and all \( \xi_i \in K(V) \).

(V3) \( \liminf_{|x| \to \infty} V(t, x) \leq -\alpha < 0 \), uniformly in \( t \).

(V4) \( V_{qq}(t, \xi_i) \) is negative definite for \( i = 1, 2, \ldots k \).

We may assume without loss of generality that \( \xi_1 = 0 \).

Remark 1.1. Notice that (V2) and (V4) imply that there are constants \( 0 < \beta_1 < \beta_2 \) and \( \eta > 0 \) such that if \( x \in B_\eta(\xi_i) \), then \( \beta_1 |x - \xi_i|^2 \leq -V(t, x) \leq \beta_2 |x - \xi_i|^2 \).

Definition 1.2. For any \( k \in \mathbb{Z}, v \in \dot{E} \), we define \( \tau_k v \) by \( \tau_k v(t) := v(t - k) \).
Notice that (V1) implies that $I(\tau_kv) = I(v)$. Thus, $I$ is invariant under a $\mathbb{Z}$ action. Notice that this also implies that $I$ does not satisfy the (PS) condition. Suppose that $K(V) = \{0, \xi\}$. In this case, it is known (see [3]) that there exist heteroclinic solutions $q_1, q_2$ of (HS) connecting 0 to $\xi$ and $\xi$ to 0. These solutions are found by minimizing $I$ over an appropriate class of functions: $q_1$ is a minimizer of $I$ over the class of functions in $\hat{E}$ starting at 0 when $t = -\infty$, and ending up at $\xi$ when $t = +\infty$. Similarly, $q_2$ is the minimizer of $I$ over the set of functions that start at $\xi$ and end at 0. In addition to these solutions of (HS), Rabinowitz also showed in [3] that given any chain of minimizers, there is an actual solution of (HS) that is “close” to the chain in an appropriate sense, provided a non-degeneracy condition holds, namely that the solutions are isolated. Recently, it has been shown in [4] that these “shadowing” solutions can be found by a minimization argument. In addition, in [5], Strobel considered the case where $V$ is periodic in all of its arguments. Thus, the set $K(V)$ is a lattice of points. We may assume that in this case, $K(V) = \mathbb{Z}^d$. We generalize our results to consider this case. In this section, we want to find heteroclinic solutions of (HS) that are not minimizers. To do this, we will introduce a minimax value $c$, as the minimax of $I$ over all paths in $\hat{E}$ that connect a minimizer $q$ of $I$ and its translate $\tau_1q$. Standard theorems (see [6]) then imply the existence of a Palais-Smale (PS) sequence $v_n$ such that $I(v_n) \to c$ as $n \to \infty$. Since $I$ does not satisfy the (PS) condition, it is necessary to investigate how such sequences “split” into convergence sequences. This is the goal of this section. Once this is done, we can use our knowledge of how (PS) sequences split to show the existence of solutions of (HS) that are not minimizers.

For future reference, we note the following

**Lemma 1.3.** [3], [5]

(i) For every $M$, there is a $C(M)$ such that if $I(q) < M$, then $\|q\|_{L^\infty} < C(M)$.

(ii) If $I(q) < \infty$, then $\lim_{t \to -\infty} q(t) := q(\infty)$ and $\lim_{t \to -\infty} q(t) := q(-\infty)$ exist, and $q(\pm \infty) \in K(V)$.

We have as a consequence of (V4) the following useful lemma:

**Lemma 1.4.** If $I(q) < \infty$, then $q - q(-\infty) \in W^{1,2}(-\infty, 0)$ and $q - q(\infty) \in W^{1,2}(0, \infty)$.

**Proof.** It suffices to show that $q - q(\infty) \in L^2([0, \infty))$. Pick $T$ so large that for $t > T$, $|q(t) - q(\infty)| < \eta$. Then, by the remark above and Lemma 1.3, we have

$$\int_T^\infty -V(t, q(t)) dt \geq \int_T^\infty \beta_1 |q(t) - q(\infty)|^2 dt,$$
and the lemma follows.

Thus, if \( q_1, q_2 \in \hat{E} \) are two functions such that \( I(q_1), I(q_2) < \infty, q_1(-\infty) = q_2(-\infty) \) and \( q_1(\infty) = q_2(\infty) \), then \( q_1 - q_2 \in E \).

**Definition 1.5.** For any \( \chi \in \hat{E} \) such that \( I(\chi) < \infty \), we define

\[
J_\chi(u) := I(\chi + u) \text{ for any } u \in E.
\]

**Proposition 1.6.** (i) \( J_\chi \in C^1(E, \mathbb{R}) \), and

\[
J_\chi'(u)\varphi = \int_{\mathbb{R}} \langle \dot{\chi} + \dot{u}, \varphi \rangle - \langle V_q(t, \chi + u), \varphi \rangle dt.
\]

(ii) \( J_\chi' = D \circ Id + K \), where \( D : E \to E' \) is the duality map, \( Id \) is the identity map on \( E \), and \( K : E \to E' \) is such that if \( \|u_n - u\|_{L^2} \to 0 \) and \( \|u_n\|_{L^\infty} \) is bounded, then \( K(u_n) \to K(u) \) in \( E' \).

With this in mind, we put \( I'(v) := J_\nu'(0) \), so

\[
I'(v)\varphi = \int_{\mathbb{R}} \langle \dot{v}, \varphi \rangle - \langle V_q(t, v), \varphi \rangle dt.
\]

Thus, if \( I(\chi) < \infty \), we have

\[
I'(u + \chi)\varphi = \int_{\mathbb{R}} \langle \dot{\chi} + \dot{u}, \varphi \rangle - \langle V_q(t, \chi + u), \varphi \rangle dt.
\]

\[
= J'_\chi(u)\varphi
\]

for all \( \varphi \in E \). Thus, \( J'_\chi(u) = I'(u + \chi) \), and we can then talk about the convergence of \( I'(v_n) \) as elements of \( E' \). Moreover, if \( I'(v) = 0 \), then \( v \) solves (HS).

**Proof.** We need to show that

\[
J'_\chi(u)\varphi = \int_{\mathbb{R}} \langle \dot{\chi} + \dot{u}, \varphi \rangle - \langle V_q(t, \chi + u), \varphi \rangle dt.
\]

Thus, we have

\[
J_\chi(u + \varphi) - J_\chi(u) - J'_\chi(u)\varphi = \int_{\mathbb{R}} \frac{1}{2} |\dot{\chi} + \dot{u} + \varphi|^2 - V(t, \chi + u + \varphi) dt
\]

\[- \int_{\mathbb{R}} \frac{1}{2} |\dot{\chi} + \dot{u}|^2 - V(t, \chi + u) - J'_\chi(u)\varphi dt
\]

\[
= \int_{\mathbb{R}} \frac{1}{2} \left( |\dot{\chi} + \dot{u} + \varphi|^2 - |\dot{\chi} + \dot{u}|^2 - 2\langle \dot{\chi} + \dot{u}, \varphi \rangle \right) dt \tag{2}
\]

\[- \int_{\mathbb{R}} (V(t, \chi + u + \varphi) - V(t, \chi + u) - \langle V_q(t, \chi + u), \varphi \rangle) dt
\]

\[
= \int_{\mathbb{R}} \frac{1}{2} |\varphi|^2 - (V(t, \chi + u + \varphi) - V(t, \chi + u) - \langle V_q(t, \chi + u), \varphi \rangle) dt
\]
Let us turn our attention to the second term in (2):
\[
\int_{\mathbb{R}} \left( V(t, \chi + u + \varphi) - V(t, \chi + u) - \langle V_q(t, \chi + u), \varphi \rangle \right) dt
\]
Because \( V \in C^2 \), we have
\[
V(t, \chi(t) + u(t) + \varphi(t)) - V(t, \chi(t) + u(t)) = \int_0^1 \frac{d}{ds} \left( V(t, u(t) + \chi(t) + s\varphi(t)) \right) ds
\]
and so we have
\[
V(t, u(t) + \chi(t) + \varphi(t)) - V(t, u(t) + \chi(t)) - \langle V_q(t, u(t) + \chi(t)), \varphi(t) \rangle
= \int_0^1 \langle V_q(t, u(t) + \chi(t) + s\varphi(t)) - V_q(t, u(t) + \chi(t)), \varphi(t) \rangle ds. \tag{3}
\]
In a similar fashion, we have that
\[
V_q(t, u(t) + \chi(t) + s\varphi(t)) - V_q(t, u(t) + \chi(t))
= \int_0^1 \frac{d}{d\alpha} \langle V_q(t, u(t) + \chi(t) + s\alpha \varphi(t)) \rangle d\alpha
= \int_0^1 V_{qq}(t, u(t) + \chi(t) + s\alpha \varphi(t)) s\varphi(t) d\alpha. \tag{5}
\]
Hence
\[
V(t, u(t) + \chi(t) + \varphi(t)) - V(t, u(t) + \chi(t)) - \langle V_q(t, u(t) + \chi(t)), \varphi(t) \rangle
= \int_0^1 \langle V_q(t, u(t) + \chi(t) + s\varphi(t)) - V_q(t, u(t) + \chi(t)), \varphi(t) \rangle ds
= \int_0^1 \int_0^1 \langle V_{qq}(t, u(t) + \chi(t) + s\alpha \varphi(t)) s\varphi(t), \varphi(t) \rangle d\alpha ds. \tag{6}
\]
But, then we have that
\[
\left| V(t, \chi(t) + u(t) + \varphi(t)) - V(t, \chi(t) + u(t)) - \langle V_q(t, \chi(t) + u(t)), \varphi(t) \rangle \right|
= \left| \int_0^1 \int_0^1 \langle V_{qq}(t, u(t) + \chi(t) + s\alpha \varphi(t)) s\varphi(t), \varphi(t) \rangle d\alpha ds \right| \tag{7}
\leq \int_0^1 \int_0^1 |V_{qq}(t, u(t) + \chi(t) + s\alpha \varphi(t))||\varphi(t)|^2 d\alpha ds.
\]

Now, notice that

\[ |u(t) + \chi(t) + s\alpha\varphi(t)| \leq \|u + \chi\|_{L^\infty} + \|\varphi\|_{L^\infty} \leq \|u + \chi\|_{L^\infty} + 1 \]

if \( \|\varphi\|_{W^{1,2}} \) is small enough. (Here, we make use of the fact that there is a constant \( C \) such that \( \|\varphi\|_{L^\infty} \leq C\|\varphi\|_{W^{1,2}} \).) Thus, for all small enough \( \varphi \), we have

\[ |V_{qq}(t, u(t) + \chi(t) + s\alpha\varphi(t))| \leq \max_{0 \leq t \leq 1, |x| \leq 1 + \|u + \chi\|_{L^\infty}} |V_{qq}(t, x)| =: M(u + \chi). \]

Notice that \( M \) is independent of \( \varphi \), supposing that \( \|\varphi\|_{W^{1,2}} \) is sufficiently small. Thus, we have

\[ \int_0^1 \int_0^1 \|V_{qq}(t, u(t) + \chi(t) + s\alpha\varphi(t))\|_{op} |\varphi(t)|^2 \, d\alpha \, ds \leq M|\varphi(t)|^2. \]

Combining all the terms, we then have that

\[ |J_\chi(u + \varphi) - J_\chi(u) - J'_\chi(u)\varphi| \]

\[ \leq \int_\mathbb{R} \frac{1}{2} |\dot{\varphi}|^2 dt + \left| \int_\mathbb{R} \left( V(t, \chi + u + \varphi) - V(t, \chi + u) - \langle V_q(t, \chi + u), \varphi \rangle \right) dt \right| \quad (8) \]

\[ \leq \int_\mathbb{R} \left( \frac{1}{2} |\dot{\varphi}|^2 + M|\varphi|^2 \right) dt \]

\[ \leq \left( \frac{1}{2} + M \right) \|\varphi\|_{W^{1,2}}^2 \text{ for all small } \varphi, \]

which implies that \( J_\chi \) is differentiable, and the derivative is what we have claimed it to be in (1). Next, we need to show that \( J'_\chi \) is continuous. We have

\[ \|J'_\chi(u) - J'_\chi(w)\| := \sup_{\|\varphi\|_{W^{1,2}} \leq 1} |J'_\chi(u)\varphi - J'_\chi(w)\varphi|. \quad (9) \]

But,

\[ \left( J'_\chi(u) - J'_\chi(w) \right)\varphi = \int_\mathbb{R} \langle \dot{\chi} + \dot{u}, \varphi \rangle - \langle V_q(t, \chi + u), \varphi \rangle dt \]

\[ - \int_\mathbb{R} \langle \dot{\chi} + \dot{w}, \varphi \rangle - \langle V_q(t, \chi + w), \varphi \rangle dt \]

\[ = \int_\mathbb{R} \langle \dot{u} - \dot{w}, \varphi \rangle - \left( (V_q(t, \chi + u) - V_q(t, \chi + w), \varphi) \right) dt \quad (10) \]

Now,

\[ \int_\mathbb{R} \langle \dot{u} - \dot{w}, \varphi \rangle dt \leq \|\dot{u} - \dot{w}\|_{L^2} \|\varphi\|_{L^2} \leq \|\dot{u} - \dot{w}\|_{L^2}, \quad (11) \]
since $\|\varphi\|_{W^{1,2}} \leq 1$. We also have

$$V_q(t, \chi + u) - V_q(t, \chi + w) = \int_0^1 \frac{d}{ds} (V_q(t, \chi + w + s(u - w))) ds$$

$$= \int_0^1 V_{qq}(t, \chi + w + s(u - w))(u - w) ds,$$

(12)

hence

$$\left| \int_{\mathbb{R}} \langle V_q(t, \chi + u) - V_q(t, \chi + w), \varphi \rangle dt \right|$$

$$= \left| \int_{\mathbb{R}} \left( \int_0^1 V_{qq}(t, \chi + w + s(u - w))(u - w) ds, \varphi \right) dt \right|$$

$$\leq \int_{\mathbb{R}} \left| \int_0^1 V_{qq}(t, \chi + w + s(u - w))(u - w) ds \right| |\varphi| dt$$

(13)

$$\leq \int_{\mathbb{R}} \left( \int_0^1 |V_{qq}(t, \chi + w + s(u - w))|_{op}(u - w) ds \right) |\varphi| dt$$

But, we have

$$|\chi(t) + w(t) + s(u(t) - w(t))| \leq \|\chi + u\|_{L^\infty} + \|u - w\|_{L^\infty}.$$

Hence

$$|\chi(t) + w(t) + s(u(t) - w(t))| \leq \|\chi + u\|_{L^\infty} + 1$$

for all $t \in \mathbb{R}$, provided $\|u - w\|_{L^\infty} < 1$. Therefore, for all $s \in [0, 1]$

$$|V_{qq}(t, \chi + w + s(u - w))| \leq \max_{0 \leq t \leq 1, |x| \leq \|\chi + w\|_{L^\infty}} |V_{qq}(t, x)| =: M$$

(14)

and so by (13) and (14)

$$\left| \int_{\mathbb{R}} \langle V_q(t, \chi + u) - V_q(t, \chi + w), \varphi \rangle dt \right| \leq \int_{\mathbb{R}} \left( \int_0^1 M|u - w| ds \right) |\varphi| dt$$

$$\leq M\|u - w\|_{L^2} \|\varphi\|_{L^2}$$

(15)

since $\|\varphi\|_{W^{1,2}} \leq 1$. Combining (11) and (15), we have

$$\|(J'_\chi(u) - J'_\chi(w))\varphi\| \leq \|\dot{u} - \dot{w}\|_{L^2} + M\|u - w\|_{L^2},$$

(16)

where $M$ depends on the $L^\infty$ norm of $u - w$. This then implies

$$\|(J'_\chi(u) - J'_\chi(w))\|_{E'} \leq C\|u - w\|_{W^{1,2}},$$
so $J'_\chi$ is locally Lipschitz. Hence $J_\chi$ is $C^1$, which proves part (i).

For part (ii), notice that

$$J'_\chi(u)\varphi = \int_\mathbb{R} \left( \langle \dot{u} + \chi, \dot{\varphi} \rangle - \langle V_q(t, \chi + u), \varphi \rangle \right) dt \quad (17)$$

$$= \int_\mathbb{R} \left( \langle \dot{u}, \dot{\varphi} \rangle + \langle u, \varphi \rangle \right) dt - \int_\mathbb{R} \left( \langle V_q(t, \chi + u), \varphi \rangle + \langle u, \varphi \rangle - \langle \dot{\chi}, \dot{\varphi} \rangle \right) dt$$

$$= \langle u, \varphi \rangle_E + (-1) \int_\mathbb{R} \left( \langle V_q(t, \chi + u), \varphi \rangle + \langle u, \varphi \rangle - \langle \dot{\chi}, \dot{\varphi} \rangle \right) dt$$

$$= \langle u, \varphi \rangle_E + K(u)\varphi,$$

where

$$K(u)\varphi := \int_\mathbb{R} \left( \langle V_q(t, \chi + u), \varphi \rangle + \langle u, \varphi \rangle - \langle \dot{\chi}, \dot{\varphi} \rangle \right) dt. \quad (18)$$

Thus, $J'_\chi(\cdot) = D \circ Id + K(\cdot)$. To prove (ii), suppose now that $u_n \to u$ in $L^2(\mathbb{R})$ and $u_n$ is bounded in $L^\infty(\mathbb{R})$. Then:

$$\left( K(u_n) - K(u) \right) \varphi = \int_\mathbb{R} \left( \langle V_q(t, \chi + u_n) - V_q(t, \chi + u), \varphi \rangle + \langle u_n - u, \varphi \rangle \right) dt,$$

so if $\|\varphi\|_{W^{1,2}} \leq 1$, we have:

$$\left| \left( K(u_n) - K(u) \right) \varphi \right| \leq \|u_n - u\|_{L^2} \|\varphi\|_{L^2} + \left| \int_\mathbb{R} \left( \langle V_q(t, \chi + u_n) - V_q(t, \chi + u), \varphi \rangle \right) dt \right|$$

$$\leq \|u_n - u\|_{L^2} + \left| \int_\mathbb{R} \left( \langle V_q(t, \chi + u_n) - V_q(t, \chi + u), \varphi \rangle \right) dt \right|. \quad (19)$$

Now, as usual, we have

$$V_q(t, \chi + u) - V_q(t, \chi + w) = \int_0^1 \frac{d}{ds} \left( V_q(t, \chi + u + s(u_n - u)) \right) ds \quad (20)$$

$$= \int_0^1 V_{qq}(t, \chi + u + s(u_n - u))(u_n - u) ds.$$

Hence

$$\left| \int_\mathbb{R} \left( \langle V_q(t, \chi + u_n) - V_q(t, \chi + u), \varphi \rangle \right) dt \right|$$

$$= \left| \int_\mathbb{R} \left( \int_0^1 V_{qq}(t, \chi + u + s(u_n - u))(u_n - u) ds, \varphi \right) dt \right|$$

$$\leq \int_\mathbb{R} \left| \int_0^1 V_{qq}(t, \chi + u + s(u_n - u))(u_n - u) ds \right| \|\varphi\| dt$$

$$\leq \int_\mathbb{R} \left( \int_0^1 \left| V_{qq}(t, \chi + u + s(u_n - u)) \right| |u_n - u| ds \right) \|\varphi\| dt. \quad (21)$$
But (as before), we know that for all $s \in [0, 1]$ and $t \in \mathbb{R}$, we have

$$|\chi(t) + u(t) + s(u_n(t) - u(t))| \leq \|\chi\|_{L^\infty} + 2\|u\|_{L^\infty} + \|u_n\|_{L^\infty} \leq \tilde{C}, \quad (22)$$

where $\tilde{C}$ depends on the $L^\infty$ norm of $\chi$, $u$ and the $L^\infty$ bounds on $u_n$. This then implies that

$$|V_{qq}(t, \chi + s + s(u_n - u))| \leq \max_{0 \leq t \leq 1, |x| \leq C} |V_{qq}(t, x)| =: M, \quad (23)$$

so combining (21) and (23), we have

$$\left| \int_{\mathbb{R}} \langle V_q(t, \chi + u_n) - V_q(t, \chi + u), \varphi \rangle dt \right|$$

$$\leq \int_{\mathbb{R}} \left( \int_0^1 |V_{qq}(t, \chi + u + s(u_n - u))| |u_n - u| ds \right) |\varphi| dt$$

$$\leq \int_{\mathbb{R}} M |u_n - u| |\varphi| dt$$

$$\leq M \|u_n - u\|_{L^2} \quad (24)$$

since $\|\varphi\|_{W^{1,2}} \leq 1$. Altogether, (19) and (24) imply that

$$\left| (K(u_n) - K(u)) \varphi \right| \leq (M + 1) \|u_n - u\|_{L^2} \quad (25)$$

for some $M$ independent of $\varphi$.

Because (25) is independent of $\varphi$, we have

$$\sup_{\|\varphi\|_{W^{1,2}} \leq 1} \left| (K(u_n) - K(u)) \varphi \right| \leq (M + 1) \|u_n - u\|_{L^2},$$

and so $K(u_n)$ converges strongly to $K(u)$.

We now prove a corollary:

**Corollary 1.7.** Suppose that $I(\chi) < \infty$, and $\{u_n\} \subset E$ and $u \in E$ are such that

(i) $\|u_n - u\|_{L^2} \to 0$ as $n \to \infty$

(ii) $J'_\chi(u_n) \to 0$ as $n \to \infty$

(iii) $u_n$ is bounded in $L^\infty$

(iv) $J'_\chi(u) = 0$

Then, $\|u_n - u\|_E \to 0$ as $n \to \infty$. 

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Proof. By Proposition 1.6, we know that
\[ J_\chi'(u) \varphi = \langle u, \varphi \rangle + K(u) \tag{26} \]
for all \( \varphi \in E \). But then by (iv)
\[
|\langle u_n - u, \varphi \rangle| \leq |J_\chi'(u_n)\varphi| + |J_\chi'(u)| + \|K(u_n) - K(u)\|_E\varphi
\]
\[
\leq \|J_\chi'(u_n)\|_{E'}\varphi_E + \|K(u_n) - K(u)\|_E\varphi_E \tag{27}
\]
for all \( \varphi \in E \) with \( \|\varphi\|_E = 1 \). By (ii), the first term in (27) goes to 0 as \( n \to \infty \). Since \( u_n \to u \) in \( L^2 \) by (i) and \( u_n \) is bounded in \( L^\infty \) by (iii), (ii) of Proposition 1.6 implies that the second term in (27) tends to 0 as \( n \to \infty \). Since
\[
\|u_n - u\|_E = \sup_{\|\varphi\|_E = 1} |\langle u_n - u, \varphi \rangle|,
\]
(27) implies the Corollary. \qed

Remark 1.8. Suppose now that we have a sequence \( \{v_n\} \subset \hat{E} \) such that:

(i) \( v_n(-\infty) = 0 \) for all \( n \), and \( v_n(\infty) = \xi_j \) for all \( n \) (i.e. all the \( v_n \) have the same asymptotics)

(ii) \( I(v_n) \to b \) and \( I'(v_n) \to 0 \), i.e. \( v_n \) is a (PS) sequence for \( I \).

Then, because of the boundedness of \( I(v_n) \), we can show that \( v_n \) is bounded in \( \hat{E} \), and therefore (along a subsequence) \( v_n \to v \) in \( \hat{E} \). If this \( v \) has the same asymptotics as the \( v_n \) and \( v \) solves (HS), then we can take \( \chi := v \). (In order to get the proper asymptotic behavior of \( v \), it will most likely be necessary to renormalize the sequence \( v_n \) in an appropriate fashion.) Notice that if we take \( u_n := v_n - v \) and \( u = 0 \), we have \( J_\chi(u_n) \to b \) and \( J_\chi'(u_n) \to 0 \). If we can show that \( \|u_n - u\|_{L^2} = \|v_n - v\|_{L^2} \to 0 \), then Corollary 1.7 implies that \( u_n \to u \) in \( E \). Thus, to determine whether or not a (PS) sequence converges, it suffices to look at the \( L^2 \) convergence of \( v_n \) to \( v \).

For the remainder of this section, we consider a sequence \( v_n \) that satisfies the conditions in the preceding remark. (One should bear in mind that \( \xi \) might possibly equal 0.) Without loss of generality, we may assume that \( I(v_n) \leq b + 1 \) for all \( n \in \mathbb{N} \).

Definition 1.9. For any \( r < \eta \) (where \( \eta \) is from Remark 1.1), we let
\[
\alpha(r) := \inf_{0 \leq t \leq 1, x \in \cup_i B_r(\xi_i)} -V(t, x)
\]
Notice that by our assumptions about the potential $V$, $\alpha(r) > 0$ for $r > 0$. We have the following fundamental lemma:

**Lemma 1.10.** [3], [5] Suppose $a < b \in \mathbb{R}$ are such that $v(t) \notin B_r(\xi_j)$ for all $t \in (a, b)$ and $j = 1, 2, \ldots, k$. Then:

$$\int_a^b \left( \frac{1}{2} |\dot{v}|^2 - V(t, v(t)) \right) dt \geq \sqrt{2\alpha(r)}|v(b) - v(a)|$$

For any $r > 0$, we let $B_r(K(V))$ denote the union of $B_r(\xi_j)$ over all $\xi_j \in K(V)$.

**Definition 1.11.** For every $n \in \mathbb{N}$, we let

$$B_n := \{ t \in \mathbb{R} | v_n(t) \in B_{\eta/2}(K(V)) \}.$$ 

Notice that $B_n$ is open, so it is the union of an at most countable number of open intervals. (Let us note that at the endpoints of these intervals, we must have $v_n \in \partial B_{\eta/2}(K(V)).$)

Suppose that $(a_n, b_n)$ is one such interval, and that $b_n - a_n \geq (b + 2)(\frac{1}{\alpha(\eta/4)})$. If there is no $t \in (a_n, b_n)$ such that $v_n(t) \in B_{\eta/4}(K(V))$, then we must have $-V(t, v_n(t)) \geq \alpha(\frac{\eta}{4})$, and so:

$$b + 1 \geq I(v_n) \geq \int_{a_n}^{b_n} -V(t, v_n(t)) dt \geq \alpha(\eta/4)(b_n - a_n) \geq (b + 2),$$

which is impossible. Thus, on any interval whose length is larger than $(\frac{b + 2}{\alpha(\eta/4)})$, there is at least one $t$ at which $v_n \in B_{\eta/4}(K(V))$. Thus, on any such interval, $v_n$ will have to travel from $\partial B_{\eta/2}(K(V))$ at the endpoints to $B_{\eta/4}(K(V))$, and so (using Lemma 1.10), on any such interval,

$$\int_{a_n}^{b_n} \left( \frac{1}{2} |\dot{v}_n|^2 - V(t, v_n(t)) \right) dt \geq \frac{\eta}{4} \sqrt{2\alpha(\frac{\eta}{4})}.$$ 

Therefore, for any $n$, there can be at most a finite number of such long intervals. Call this number $l(n)$. Moreover, the number $l(n)$ of such intervals must be bounded independent of $n$. Now, let us consider the maximal intervals $(-\infty, \bar{t}_{n_1}), (\bar{t}_{n_2}, \bar{t}_{n_2}), \ldots, (\bar{t}_{n_{l(n)}}, \infty)$ in $B_n$ that have the additional property that their lengths go to $\infty$ as $n \to \infty$. Since the number of such intervals $l(n)$ is bounded independent of $n$ for any PS sequence, we can assume that $l(n)$ is the same for all large enough $n$.

**Lemma 1.12.** $\bar{t}_{n_{i+1}} - \bar{t}_{n_i}$ is bounded independently of $n, i$.

**Proof.** Suppose not. Then, we have

$$(\bar{t}_{n_i}, \bar{t}_{n_{i+1}}) = (B_n \cap (\bar{t}_{n_i}, \bar{t}_{n_{i+1}})) \cup \{ t \in (\bar{t}_{n_i}, \bar{t}_{n_{i+1}}) | v_n(t) \notin B_{\eta/2}(K(V)) \}$$

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By Lemma 1.10, the measure of the second term must be bounded, since $I(v_n) \leq b + 1$ for all $n$. Thus,

$$\left| \left( B_n \cap (\bar{t}_{n_1}, \bar{t}_{n+1}) \right) \right| \to \infty$$

(28)
as $n \to \infty$, where $|A|$ denotes the measure of a set $A$. Now,

$$\left( B_n \cap (\bar{t}_{n_1}, \bar{t}_{n+1}) \right) = \bigcup_{i=1}^{j(n)} \left( a^n_i, b^n_i \right),$$

(29)
where the $(a^n_i, b^n_i)$ are maximal intervals in which $v_n(t) \in B_{\eta/2}(K(V))$. Notice that because of the maximality of $(a^n_i, b^n_i)$, $v_n \in \partial B_{\eta/2}(K(V))$ at the endpoints of these intervals. Now,

$$b^n_i - a^n_i \text{ is bounded independent of } n$$

(30)
because $(-\infty, \bar{t}_{n_1}), (\bar{t}_{n_2}, \bar{t}_{n_1}), \ldots, (\bar{t}_{n_1}, \infty)$ are the only subintervals of $B_n$ whose lengths are unbounded in $n$. Thus, in order for (28) to be true, we must have $j(n) \to \infty$ as $n \to \infty$. Using Lemma 1.10, we cannot have $v_n(t) \notin B_{\eta/4}(K(V))$ for all $t \in (B_n \cap (\bar{t}_{n_1}, \bar{t}_{n+1}))$ because of the boundedness of $I(v_n)$. Therefore, (using $\sqcup$ to denote a disjoint union),

$$\left( B_n \cap (\bar{t}_{n_1}, \bar{t}_{n+1}) \right) = \left( \sqcup_{i=1}^{j_G(n)} (\bar{a}^n_i, \bar{b}^n_i) \right) \sqcup \left( \sqcup_{i=1}^{j_B(n)} (\bar{c}^n_i, \bar{d}^n_i) \right)$$

(31)
where $j_G(n)$ is the number of intervals where $v_n(t) \in B_{\eta/4}(K(V))$ for some $t \in (\bar{a}^n_i, \bar{b}^n_i)$, and $j_B(n)$ is the number of intervals where $v_n$ avoids $B_{\eta/4}(K(V))$. By (29) and (31), we must have

$$j_G(n) + j_B(n) = j(n) \to \infty$$

(32)
as $n \to \infty$. Thus, by (32), at least one of $j_G(n), j_B(n) \to \infty$ as $n \to \infty$.

Now, on each interval $(\bar{a}^n_i, \bar{b}^n_i)$, $v_n$ goes from $\partial B_{\eta/2}(K(V))$ at the endpoints to $B_{\eta/4}(K(V))$ somewhere inside. Thus by Lemma 1.10, on each of these $j_G(n)$ intervals, we get a contribution of at least $\frac{\eta}{4} \sqrt{2\alpha(\eta/4)}$ to $I(v_n)$. Since $I(v_n)$ is bounded, $j_G(n) \leq C_1$ for some $C_1$, and so by (30),

$$\left| \sqcup_{i=1}^{j_G(n)} (\bar{a}^n_i, \bar{b}^n_i) \right| \leq C_2,$$

(33)
for some $C_2$. Thus, by (32), $j_B(n) \to \infty$.

By Lemma 1.10, the length of any interval $(\bar{c}^n_i, \bar{d}^n_i)$ where $v_n$ avoids $B_{\eta/4}(K(V))$ must be bounded. By (28) and (33), we have

$$\left| \sqcup_{i=1}^{j_B(n)} (\bar{c}^n_i, \bar{d}^n_i) \right| \to \infty.$$
But then Definition 1.9 and (34) imply that

\[ I(v_n) \geq \sum_{i=1}^{j_0(n)} \int_{c_i^{n}}^{d_i^{n}} \alpha \left( \frac{\eta}{4} \right) dt = \alpha \left( \frac{\eta}{4} \right) \sum_{i=1}^{j_0(n)} (d_i^{n} - c_i^{n}) \to \infty \]

as \( n \to \infty \), which contradicts the fact that \( I(v_n) \) is bounded.

\[ \square \]

**Proposition 1.13.** Suppose that there are only two such intervals: \((-\infty, \bar{t}_{n_1}) \) and \((\bar{t}_{n_2}, \infty) \). Then, there is a sequence \( k_n \in \mathbb{Z} \) such that

(i) \( I(\tau_{k_n}v_n) \to b \), and \( I'(\tau_{k_n}v_n) \to 0 \)

(ii) There is a \( v \in \hat{E} \) solving \((HS)\) such that \( \|\tau_{k_n}v_n - v\|_{W^{1,2}(R)} \to 0 \)

**Proof.** Pick \( k_n \in \mathbb{Z} \) such that \( \bar{t}_{n_1} + k_n \in [0, 1) \). We have \( \tau_{k_n}v_n(\bar{t}_{n_1} + k_n) = v_n(\bar{t}_{n_1}) \in \partial B_{\eta/2}(K(V)) \), and \( I(\tau_{k_n}v_n) = I(v_n) \) (because of the \( \mathbb{Z} \) action). Moreover, \( \|I'(\tau_{k_n}v_n)\| = \|I'(v_n)\| \). Thus, (i) is proven.

Because of the choice of \( k_n \), we have \( 0 \leq \bar{t}_{n_1} + k_n \), hence \( \tau_{k_n}v_n(0) \in B_{\eta/2}(0) \) for all \( n \). Because \( b + 1 \geq I(\tau_{k_n}v_n) \), we also have \( \|\tau_{k_n}v_n\|_{L^2} \) bounded. Thus \( \tau_{k_n}v_n \) is bounded in \( \hat{E} \), and so (along a subsequence) \( \tau_{k_n}v_n \rightharpoonup v \) for some \( v \in \hat{E} \). In particular, this means that \( \tau_{k_n}v_n \to \hat{v} \) in \( L^2 \). In addition, since \( I(v_n) \leq b + 1 \) for all \( n \) and \( v_n(-\infty) = 0 \) for all \( n \), we know that \( \tau_{k_n}v_n \) is bounded in \( L^\infty \). Together, these two facts imply that \( \tau_{k_n}v_n \) converges to \( v \) in \( L^\infty_{loc} \). By the weak lower semi-continuity of \( I \) (see [2] and [3]), \( I(v) \leq b \). Next, we show that \( v \) solves \((HS)\). To do this, it suffices to show that

\[ I'(v)\varphi = \int_R \langle \dot{v}, \varphi \rangle - \langle V_q(t, v), \varphi \rangle \, dt = 0 \tag{35} \]

for all \( \varphi \in C_c^\infty(R) \). Fix \( \varphi \in C_c^\infty(R) \). Then, we know that

\[ I'(\tau_{k_n}v_n)\varphi = \int_R \langle \tau_{k_n}\dot{v}_n, \varphi \rangle - \langle V_q(t, v_n), \varphi \rangle \, dt \to 0 \tag{36} \]

as \( n \to \infty \). Since \( \tau_{k_n}v_n \to v \) in \( L^\infty_{loc} \) and \( \varphi \) has compact support, we know that

\[ \int_R \langle V_q(t, \tau_{k_n}v_n), \varphi \rangle \, dt \to \int_R \langle V_q(t, v), \varphi \rangle \, dt \tag{37} \]

as \( n \to \infty \). Moreover, since \( \tau_{k_n}\dot{v}_n \to \hat{v} \) in \( L^2 \), we know that

\[ \int_R \langle \tau_{k_n}\dot{v}_n, \varphi \rangle \, dt \to \int_R \langle \hat{v}, \varphi \rangle \, dt \tag{38} \]
as } n \to \infty. \text{ Combining (36) - (38), we see that }

\begin{equation}
0 = \lim_{n \to \infty} I'(\tau_n v_n) \varphi = \lim_{n \to \infty} \left( \int_{\mathbb{R}} \langle \tau_n \dot{v}_n, \varphi \rangle - \langle V_q(t, \tau_n v_n), \varphi \rangle \, dt \right) = \int_{\mathbb{R}} \langle \dot{v}, \varphi \rangle - \langle V_q(t, v), \varphi \rangle \, dt \tag{39}
= I'(v) \varphi.
\end{equation}

Since (39) holds for all } \varphi \in C_c^\infty(\mathbb{R}), v \text{ solves (HS).}

To get (ii), we want to apply Corollary 1.7. Let } \chi := v, u_n := \tau_n v_n - v \text{ and } u := 0. \text{ Notice that } J'_\chi(u_n) = I'(\tau_n v_n) \to 0 \text{ as } n \to \infty. \text{ Moreover, we know that } v_n \text{ is bounded in } L^\infty, \text{ since } I(v_n) \leq b + 1, \text{ hence } \|u_n\|_{L^\infty} \leq \|v_n\|_{L^\infty}. \text{ Moreover, since } v \text{ solves (HS), } 0 = I'(v) = J'_\chi(u).

Thus, to apply Corollary 1.7, we need to show that

\begin{equation}
\|u_n - u\|_{L^2} = \|\tau_n v_n - v\|_{L^2} \to 0 \tag{40}
\end{equation}
as } n \to \infty. \text{ Once we have (40), Corollary 1.7 implies that }

\begin{equation}
\|u_n - u\|_{E} = \|\tau_n v_n - v\|_{E} \to 0 \tag{41}
\end{equation}
as } n \to \infty, \text{ which implies (ii) of Proposition 1.13.}

First, we show that } v \neq \text{ constant. Passing to a subsequence, we must have } \bar{t}_{n_1} + k_n \to \bar{t} \in [0, 1]. \text{ We claim that } \tau_n v_n(\bar{t}_{n_1} + k_n) \to v(\bar{t}). \text{ If so, then since } \tau_n v_n(\bar{t}_{n_1} + k_n) = v_n(\bar{t}_{n_1}) \in \partial B_{\eta/2}(K(V)) \text{ for all } n, \text{ we will have } v(\bar{t}) \in \partial B_{\eta/2}(K(V)). \text{ This implies } v \neq \text{ constant, since } I(v) < \infty, \text{ and the only constant functions for which } I(v) < \infty \text{ are when } v \equiv \xi \text{ for some } \xi \in K(V). \text{ To prove our claim, note that }

\begin{equation}
|v(\bar{t}) - \tau_n v_n(\bar{t}_{n_1} + k_n)| \leq |v(\bar{t}) - \tau_n v_n(\bar{t})| + |\tau_n v_n(\bar{t}) - \tau_n v_n(\bar{t}_{n_1} + k_n)|.
\end{equation}
The first term goes to 0 because of the } L^\infty \text{ convergence of } \tau_n v_n \text{ to } v. \text{ The second term goes to 0 because } \{\tau_n v_n\} \text{ is equicontinuous (by the } L^2 \text{ bound on the derivatives).}

Next, we claim that } v(t) \in B_{\eta/2}(0) \text{ for all } t < 0. \text{ To see this, note that (by the choice of } k_n \text{ and the assumption about the behavior of our sequence at } - \infty) \tau_n v_n(t) \in B_{\eta/2}(0) \text{ for all } t < 0. \text{ Thus, by pointwise convergence, we must have } v(t) \in B_{\eta/2}(0) \text{ for all } t < 0. \text{ Therefore, (since } I(v) < \infty \text{) we must have } v(-\infty) = 0 = \tau_n v_n(-\infty) \text{ for all } n.

Suppose now that } v_n(\infty) = \xi \text{ for all } n \text{ and some } \xi \in K(V). \text{ We want to show that } v(t) \in B_{\eta/2}(\xi) \text{ for all sufficiently large } t, \text{ which will imply (as above) that } v(\infty) = \xi. \text{ We know from Lemma 1.12 that } (\bar{t}_{n_2} - \bar{t}_{n_1}) \text{ is bounded, so } \bar{t}_{n_2} - \bar{t}_{n_1} \to t^*. \text{ Because } \bar{t}_{n_1} + k_n \to \bar{t},
we have \( t_n + k_n = \bar{t}_n - \bar{t}_n + k_n \rightarrow \bar{t} + t^* \). Suppose now that \( t > \bar{t} + t^* + 1 \). Then, for sufficiently large \( n \), we have \( t_n + k_n < \bar{t} + t^* + 1 < t \), hence \( t_n < t - k_n \) for all large \( n \). But then \( v_n(t - k_n) \in B_{\eta/2}(\xi) \), so \( \tau_{t_n} v_n(t) \in B_{\eta/2}(\xi) \) for all large \( n \), and so \( v(t) \in \overline{B_{\eta/2}(\xi)} \). Thus, \( v(\infty) = \xi \). Notice that the preceding two paragraphs prove the following

**Proposition 1.14.** There is an \( N \) such that if \( n > N \), then

(i) \( \tau_{t_n} v_n(t) \in B_{\eta/2}(0) \) for \( t < 0 \)

(ii) \( \tau_{t_n} v_n(t) \in B_{\eta/2}(\xi) \) for \( t > \bar{t} + t^* + 1 \)

So, \( \tau_{t_n} v_n \) and \( v \) have the same asymptotics. To prove part (ii), by Corollary 1.7, it suffices to show that \( \| \tau_{t_n} v_n - v \|_{L^2} \to 0 \). If this is not the case, there must be a \( \delta > 0 \) such that on a subsequence (which we relabel)

\[
\| \tau_{t_n} v_n - v \|_{L^2} \geq \delta.
\] (42)

For any \( \rho < \frac{\eta}{4} \), let

\[
\bar{t}_n(\rho) := \inf\{ t \mid v(s) \in B_\rho(0) \text{ for } s > t \},
\]

\[
\bar{t}_n(\rho) := \inf\{ t \mid v(s) \in B_\rho(\xi) \text{ for } s > t \}.
\] (43)

Notice that we have \( \bar{t}_n(\rho) \to -\infty \) and \( \bar{t}_n(\rho) \to \infty \) as \( \rho \to 0 \). Next, let \( A(\rho) := (\bar{t}_n(\rho), \bar{t}_n(\rho)) \), and \( B(\rho) := \mathbb{R} - A(\rho) \). For a fixed \( \rho > 0 \), the local uniform convergence of \( \tau_{t_n} v_n \) to \( v \) implies that

\[
\| \tau_{t_n} v_n - v \|_{L^2(A(\rho))} \to 0
\] (44)
as \( n \to \infty \). We will now show that there is a \( \rho > 0 \) such that for all \( n > N(\rho) \), one has

\[
\| \tau_{t_n} v_n - v \|_{L^2(-\infty, \bar{t}_n(\rho))} < \frac{\delta}{4}
\] (45)

and

\[
\| \tau_{t_n} v_n - v \|_{L^2(\bar{t}_n(\rho), \infty)} < \frac{\delta}{4}.
\] (46)

Now by Lemma 1.4, we may pick \( \rho \) suitably small that \( \| v \|_{L^2(-\infty, \bar{t}_n(\rho))} < \delta/8 \) and \( \| v - \xi \|_{L^2(\bar{t}_n(\rho), \infty)} < \delta/8 \). In order to show (45) and (46), it suffices to show that there is an \( N(\rho) \) such that for \( n > N(\rho) \), one has

\[
\| \tau_{t_n} v_n \|_{W^{1,2}(-\infty, \bar{t}_n(\rho))} < \frac{\delta}{8}
\] (47)
and

\[ \| \tau_k v_n - \xi \|_{W^{1,2}(\bar{t}(\rho), \infty)} < \frac{\delta}{8} \]  

(48)

To do this, we need the following proposition

**Proposition 1.15.** For any fixed \( \rho \), there is an \( N(\rho) \) such that for \( n > N(\rho) \)

(i) \( \tau_k v_n(t) \in B_{2\rho}(0) \) for \( t < \bar{t}(\rho) \)

(ii) \( \tau_k v_n(t) \in B_{2\rho}(\xi) \) for \( t > \bar{t}(\rho) \)

Assuming Proposition 1.15, let us continue with the proof of Proposition 1.13. We have

\[
\frac{1}{2} \| \tau_k \dot{v}_n \|_{L^2(-\infty, \bar{t}(\rho))}^2 = \int_{-\infty}^{\bar{t}(\rho)} \frac{1}{2} |\tau_k \dot{v}_n|^2 dt
\]

\[
= \frac{1}{2} \int_{-\infty}^{\bar{t}(\rho)} \left( |\tau_k \dot{v}_n|^2 - \langle V_q(t, \tau_k v_n), \tau_k v_n \rangle \right) dt
\]

\[
+ \int_{-\infty}^{\bar{t}(\rho)} \left( \frac{1}{2} \langle V_q(t, \tau_k v_n), \tau_k v_n \rangle - V(t, \tau_k v_n) \right) dt + \int_{-\infty}^{\bar{t}(\rho)} V(t, \tau_k v_n) dt
\]

(49)

and so

\[
\frac{1}{2} \| \tau_k \dot{v}_n \|_{L^2(-\infty, \bar{t}(\rho))}^2 + \int_{-\infty}^{\bar{t}(\rho)} V(t, \tau_k v_n) dt
\]

\[
= \frac{1}{2} \int_{-\infty}^{\bar{t}(\rho)} \left( |\tau_k \dot{v}_n|^2 - \langle V_q(t, \tau_k v_n), \tau_k v_n \rangle \right) dt
\]

\[
+ \int_{-\infty}^{\bar{t}(\rho)} \left( \frac{1}{2} \langle V_q(t, \tau_k v_n), \tau_k v_n \rangle - V(t, \tau_k v_n) \right) dt
\]

(50)

By Proposition 1.14, Remark 1.1 and the fact \( \tau_k v_n(-\infty) = 0 \), we have

\[
\beta_1 \int_{-\infty}^{\bar{t}(\rho)} |\tau_k v_n|^2 \leq \int_{-\infty}^{\bar{t}(\rho)} -V(t, \tau_k v_n) dt.
\]

(51)

Hence, by (50) and (51), we have

\[
\frac{1}{2} \| \tau_k \dot{v}_n \|_{L^2(-\infty, \bar{t}(\rho))}^2 + \beta_1 \int_{-\infty}^{\bar{t}(\rho)} |\tau_k v_n|^2 dt
\]

\[
\leq \frac{1}{2} \int_{-\infty}^{\bar{t}(\rho)} \left( |\tau_k \dot{v}_n|^2 - \langle V_q(t, \tau_k v_n), \tau_k v_n \rangle \right) dt
\]

\[
+ \int_{-\infty}^{\bar{t}(\rho)} \left( \frac{1}{2} \langle V_q(t, \tau_k v_n), \tau_k v_n \rangle - V(t, \tau_k v_n) \right) dt.
\]

(52)
Now, by the assumptions about our potential $V$, if $|x| \leq 2\rho$, then

$$\left| \frac{1}{2} \langle V_q(t, x), x \rangle - V(t, x) \right| \leq \epsilon(\rho)|x|^2,$$

where $\epsilon(\rho) \to 0$ as $\rho \to 0$. By Proposition 1.15, we have for all $n > N(\rho)$ that $\tau_n v_n(t) \in B_{2\rho}(0)$ for $t < \tilde{t}(\rho)$, and so

$$\left| \frac{1}{2} \langle V_q(t, \tau_n v_n(t)), \tau_n v_n(t) \rangle - V(t, \tau_n v_n(t) \rangle \right| \leq \epsilon(\rho)|\tau_n v_n(t)|^2$$

for all $n > N(\rho)$ and $t < \tilde{t}(\rho)$. Thus, if we take $\rho$ sufficiently small that $\epsilon(\rho) < \frac{1}{2}\beta_1$, then using (53), we can move the last term on the right in (52) to the left and absorb to get

$$\int_{-\infty}^{\tilde{t}(\rho)} \min(\beta_1, 1)\|\tau_n v_n\|_{W^{1,2}}^2 \leq \frac{1}{2} \int_{-\infty}^{\tilde{t}(\rho)} \left( |\tau_n \dot{v}_n|^2 - \langle V_q(t, \tau_n v_n), \tau_n v_n \rangle \right) dt$$

for all $n > N(\rho)$. Now, we want to show that the right hand side is $< \frac{\delta}{8}$. Let

$$\psi_n(t) := \begin{cases} 
\tau_n v_n(t) & \text{for } t < \tilde{t}(\rho) \\
(t(\rho) + 1 - t) \tau_n v_n(\tilde{t}(\rho)) & \text{for } \tilde{t}(\rho) \leq t \leq \tilde{t}(\rho) + 1 \\
0 & \text{for } t > \tilde{t}(\rho) + 1
\end{cases}$$

We claim now that $\psi_n$ is bounded in $W^{1,2}$ independently of $n$ and $\rho$. To see this, note that by (43) and (51)

$$\|\psi_n\|_{W^{1,2}}^2 = \int_{-\infty}^{\tilde{t}(\rho)} |\tau_n \dot{v}_n|^2 + |\tau_n v_n|^2 dt + \int_{\tilde{t}(\rho)}^{\tilde{t}(\rho) + 1} \left( |\tau_n v_n(t(\rho))|^2 + ((t(\rho) + 1) - t)^2 |\tau_n v_n(t(\rho))|^2 \right) dt$$

$$\leq \int_{-\infty}^{\tilde{t}(\rho)} \left( |\tau_n \dot{v}_n|^2 - \frac{1}{\beta_1} V(t, \tau_n v_n) \right) dt + 2|\tau_n v_n(\tilde{t}(\rho))|^2$$

$$\leq \left( 2 + \frac{1}{\beta_1} \right) I(\tau_n v_n) + \eta^2 < \left( 2 + \frac{1}{\beta_1} \right) (b + 1) + \eta^2$$

since $\rho < \eta/4$. Notice that this implies that $I'(\tau_n v_n)\psi_n \to 0$ as $n \to \infty$, since

$$|I'(\tau_n v_n)| \leq \|I'(\tau_n v_n)\| \|\psi_n\|_{W^{1,2}} \leq C\|I'(\tau_n v_n)\|.$$  

But

$$I'(\tau_n v_n)\dot{\psi}_n = \int_{-\infty}^{\tilde{t}(\rho)} \left( |\tau_n \dot{v}_n|^2 - \langle V_q(t, \tau_n v_n), \tau_n v_n \rangle \right) dt$$

$$+ \int_{\tilde{t}(\rho)}^{\tilde{t}(\rho) + 1} \langle \tau_n \dot{v}_n(t), \tau_n v_n(\tilde{t}(\rho)) \rangle dt$$

$$+ \int_{\tilde{t}(\rho)}^{\tilde{t}(\rho) + 1} -\langle V_q(t, \tau_n v_n), ((\tilde{t}(\rho) + 1) - t) \tau_n v_n(\tilde{t}(\rho)) \rangle dt.$$
Let us turn our attention to the last two terms in (57). By (43), we have
\[
\left| \int_{\mathcal{E}^{(\rho)}_n} \langle \tau_{k_n} \dot{v}_n(t), \tau_{k_n} v_n(t(\rho)) \rangle \right| \leq \left( \int_{\mathcal{E}^{(\rho)}_n} |\tau_{k_n} \dot{v}_n| |\tau_{k_n} v_n(t(\rho))| \right) dt \tag{58}
\]
\[
\leq 2\rho \left( \int_{\mathbb{R}} |\tau_{k_n} \dot{v}_n|^2 dt \right)^{\frac{1}{2}} \leq C \rho
\]
and
\[
\left| \int_{\mathcal{E}^{(\rho)}_n} \left( \langle V_q(t, \tau_{k_n} v_n), (t(\rho) + 1) - t \rangle \tau_{k_n} v_n(t(\rho)) \right) dt \right|
\leq \max_{0 \leq t \leq 1, |x| \leq \eta} |V_q(t, x)| \|\tau_{k_n} v_n(t(\rho))| \]
\leq K \rho \quad \text{(because of Prop. 1.15)} \tag{59}
\]
Thus, combining (58) and (59), the second term of (57) is bounded by \(\tilde{C} \ast \rho\) for some \(\tilde{C}\) independent of \(\rho\). Because of (54) and (57), we have shown the following:
\[
\min(1, \beta_1) \|\tau_{k_n} v_n\|_{W^{1,2}(\mathbb{R}^2)} \leq \int_{\mathcal{E}^{(\rho)}_n} \left| \langle \tau_{k_n} \dot{v}_n \rangle^2 - \langle V_q(t, \tau_{k_n} v_n), \tau_{k_n} v_n \rangle \right| dt
\]
\[
= I'(\tau_{k_n} v_n) \psi_n - \int_{\mathcal{E}^{(\rho)}_n} \langle \tau_{k_n} \dot{v}_n(t), \tau_{k_n} v_n(t(\rho)) \rangle dt
\]
\[
- \int_{\mathcal{E}^{(\rho)}_n} \langle V_q(t, \tau_{k_n} v_n), (t(\rho) + 1) - t \rangle \tau_{k_n} v_n(t(\rho)) \rangle dt
\]
By (56), \(I'(\tau_{k_n} v_n) \psi_n \to 0\) as \(n \to \infty\). Moreover, the last two terms in (60) are bounded by \(\tilde{C} \rho\). Thus, taking \(\rho\) suitably small, we have for all sufficiently large \(n\) that \(\|\tau_{k_n} v_n\|_{W^{1,2}(\mathbb{R}^2)} < \frac{\delta}{8}\).

To finish the proof of Proposition 1.13, it suffices then to show that there is a \(\rho\) such that for all \(n > N(\rho)\) we have \(\|\tau_{k_n} v_n - \xi\|_{W^{1,2}(\mathbb{R}^2)} \leq \frac{\delta}{8}\). This is proved in an analogous fashion to the previous case, with a some added complication. By Lemma 1.4 we have \(\tau_{k_n} v_n - \xi \in L^2(0, \infty)\) because \(\tau_{k_n} v_n(\infty) = \xi\) and \(I(\tau_{k_n} v_n)\) is finite. By Proposition 1.14 we have for all sufficiently large \(t\) and \(n\) that \(\tau_{k_n} v_n(t) \in B_{\eta/2}(\xi)\), and so by Remark 1.1, we have
\[
\beta_1 |\tau_{k_n} v_n(t) - \xi|^2 \leq -V(t, \tau_{k_n} v_n(t)). \tag{61}
\]
As before we have
\[
\frac{1}{2} \|\tau_{k_n} \dot{v}_n\|_{L^2(\mathbb{R}^2, \infty)}^2 = \frac{1}{2} \int_{\mathcal{E}^{(\rho)}_n} \left| \langle \tau_{k_n} \dot{v}_n \rangle^2 - \langle V_q(t, \tau_{k_n} v_n), \tau_{k_n} v_n - \xi \rangle \right| dt
\]
\[
+ \int_{\mathcal{E}^{(\rho)}_n} \left( \frac{1}{2} \langle V_q(t, \tau_{k_n} v_n), \tau_{k_n} v_n - \xi \rangle - V(t, \tau_{k_n} v_n) \right) dt \tag{62}
\]
\[
+ \int_{\mathcal{E}^{(\rho)}_n} V(t, \tau_{k_n} v_n) dt
\]
But, by (61), we know that
\[ \int_{\bar{t}(\rho)}^{\infty} V(t, \tau_{k_n} t v_n)\,dt \leq -\beta_1 \int_{\bar{t}(\rho)}^{\infty} |\tau_{k_n} v_n - \xi|^2 \,dt. \]  
(63)

Hence, we have
\[ \frac{1}{2} \|\tau_{k_n} \dot{v}_n\|_{L^2(\bar{t}(\rho), \infty)}^2 + \beta_1 \|\tau_{k_n} v_n - \xi\|_{L^2(\bar{t}(\rho), \infty)}^2 \leq \frac{1}{2} \left( \int_{\bar{t}(\rho)}^{\infty} |\tau_{k_n} \dot{v}_n|^2 - \langle V_q(t, \tau_{k_n} v_n), \tau_{k_n} v_n - \xi \rangle \,dt \right) \]
\[ + \int_{\bar{t}(\rho)}^{\infty} \left( \frac{1}{2} \langle V(t, \tau_{k_n} v_n), \tau_{k_n} v_n - \xi \rangle - V(t, \tau_{k_n} v_n) \right) \,dt \]  
(64)

Now, if \(|x - \xi| \leq 2\rho\), then
\[ \left| \frac{1}{2} \langle V_q(t, x), x - \xi \rangle - V(t, x) \right| \leq \epsilon(\rho)|x - \xi|^2, \]  
(65)

where \(\epsilon(\rho) \to 0\) as \(\rho \to 0\). Therefore, the last term on the right in (64) is then bounded by \(\epsilon(\rho)\|\tau_{k_n} v_n - \xi\|_{L^2(\bar{t}(\rho), \infty)}\). Taking \(\rho\) sufficiently small, we have (as before)
\[ \frac{1}{2} \min(1, \beta_1) \|\tau_{k_n} v_n - \xi\|_{W^{1,2}(\bar{t}(\rho), \infty)}^2 \leq \frac{1}{2} \left( \int_{\bar{t}(\rho)}^{\infty} |\tau_{k_n} \dot{v}_n|^2 - \langle V_q(t, \tau_{k_n} v_n), \tau_{k_n} v_n - \xi \rangle \,dt \right) \]  
(66)

We proceed as before: Let
\[ \psi_n(t) := \begin{cases} \tau_{k_n} v_n(t) - \xi & \text{for } t > \bar{t}(\rho) \\ (t - (\bar{t}(\rho) - 1))(\tau_{k_n} v_n(\bar{t}(\rho)) - \xi) & \text{for } \bar{t}(\rho) - 1 \leq t \leq \bar{t}(\rho) \\ 0 & \text{for } t < \bar{t}(\rho) - 1 \end{cases} \]

We have that \(\|\psi_n\|_{W^{1,2}}\) is bounded independent of \(n\) and \(\rho\). The proof of this fact is the same as before, bearing in mind that
\[ |\tau_{k_n} v_n(t) - \xi|^2 \leq -\frac{1}{\beta_1} V(t, \tau_{k_n} v_n(t)) \]
for \(t > \bar{t}(\rho)\). Thus, \(I'(\tau_{k_n} v_n) \psi_n \to 0\) as \(n \to \infty\). We have
\[ I'(\tau_{k_n} v_n) \psi_n = \int_{\bar{t}(\rho)}^{\infty} \left( |\tau_{k_n} \dot{v}_n|^2 - \langle V_q(t, \tau_{k_n} v_n), \tau_{k_n} v_n - \xi \rangle \right) \,dt \]
\[ + \int_{\bar{t}(\rho)}^{\bar{t}(\rho) - 1} \langle \tau_{k_n} \dot{v}_n, \tau_{k_n} v_n(\bar{t}(\rho)) - \xi \rangle \,dt \]
\[ - \int_{\bar{t}(\rho) - 1}^{\bar{t}(\rho)} \langle V_q(t, \tau_{k_n} v_n), (t - \bar{t}(\rho) + 1)(\tau_{k_n} v_n(\bar{t}(\rho)) - \xi) \rangle \,dt \]  
(67)
We turn our attention to the second term. Using Proposition 1.15 and (43), we have

\[
\left| \int_{\bar{t}(\rho)}^{\bar{t}(\rho)-1} \langle \tau_{\kappa_n} \dot{v}_n, \tau_{\kappa_n} v_n(\bar{t}(\rho)) - \xi \rangle dt \right| \leq |\tau_{\kappa_n} v_n(\bar{t}(\rho)) - \xi| \left( \int_{\bar{t}(\rho)-1}^{\bar{t}(\rho)} |\tau_{\kappa_n} \dot{v}_n|^2 dt \right)^{\frac{1}{2}}
\]

\[
\leq (2\rho) \|\tau_{\kappa_n} \dot{v}_n\|_{L^2}
\]

\[
\leq (2\rho) (2I(\tau_{\kappa_n} v_n))
\]

\[
\leq C\rho \text{ for } C \text{ independent of } \rho \text{ and } n
\]

Similarly

\[
\left| \int_{\bar{t}(\rho)}^{\bar{t}(\rho)-1} \langle V_q(t, \tau_{\kappa_n} v_n), (t - \bar{t}(\rho) + 1) (\tau_{\kappa_n} v_n(\bar{t}(\rho)) - \xi) \rangle dt \right|
\]

\[
\leq \max_{0 \leq t \leq 1, |x| \leq \eta} |V_q(t, x)| \|\tau_{\kappa_n} v_n(\bar{t}(\rho)) - \xi\|
\]

\[
\leq (2\rho) \max_{0 \leq t \leq 1, |x| \leq \eta} |V_q(t, x)|
\]

\[
\leq \tilde{C}\rho \text{ (for } \tilde{C} \text{ independent of } n \text{ and } \rho)
\]

The rest of the argument runs as before.

Thus, we must have \(\|\tau_{\kappa_n} v_n - v\|_{L^2} \to 0\), and so by Corollary 1.7, we have (ii) of the lemma.

Next, we have to prove Proposition 1.15. To do this, we will need some \(L^\infty\) estimates on a Palais-Smale sequence.

**Lemma 1.16.** Suppose that \(v_n\) is a sequence in \(\hat{E}\) such that \(I(v_n) \to b > 0\) and \(I'(v_n) \to 0\). Moreover, suppose that \(v_n(-\infty) = \xi_j\) for all \(n\), and \(v_n(\infty) = \xi_i\) for all \(n\). Then for all sufficiently large \(n\), \(\|v_n - \xi_j\|_{L^\infty} \geq \eta/2\).

**Proof.** Notice that the result is true if \(\xi_i \neq \xi_j\), since \(|\xi_i - \xi_j| \geq \eta\). Thus, suppose that \(\xi_i = \xi_j\), and that \(\|v_n - \xi_j\|_{L^\infty} < \eta/2\) for all large \(n\). We will now derive a contradiction. We have

\[
b + 1 \geq I(v_n) = \int_{\mathbb{R}} \left( \frac{1}{2} |\dot{v}_n|^2 - V(t, v_n) \right) dt
\]

\[
\geq \int_{\mathbb{R}} \left( \frac{1}{2} |\dot{v}_n|^2 + \beta_1 |v_n - \xi_j|^2 \right) dt
\]

\[
\geq \min \left\{ \frac{1}{2}, \beta_1 \right\} \int_{\mathbb{R}} (|\dot{v}_n|^2 + |v_n - \xi_j|^2) dt
\]

Thus,

\[
\|v_n - \xi_j\|_{W^{1,2}} \leq \frac{b + 1}{\min \left\{ \frac{1}{2}, \beta_1 \right\}}
\]
But then \( I'(v_n)(v_n - \xi_j) \to 0 \) as \( n \to 0 \). But we have

\[
I'(v_n)(v_n - \xi_j) = \int_{\mathbb{R}} (|\dot{v}_n|^2 - (V_q(t, v_n^1), v_n^1 - \xi_j)) \, dt
\]

\[
\geq \int_{\mathbb{R}} (|\dot{v}_n|^2 + \beta_1|v_n - \xi_j|^2) \, dt.
\]

In (71), we use the fact that \( \|v_n - \xi_j\|_{L^\infty} < \eta/2 \). Therefore, we have

\[
I'(v_n)(v_n - \xi_j) \geq \min(1, \beta_1)\|v_n - \xi_j\|^2_{W^{1,2}}
\]

Thus, we must have \( \|v_n - \xi_j\|_{W^{1,2}} \to 0 \) as \( n \to 0 \). But this means (recalling the definition of \( J_{\chi}(u) \) for \( u \in W^{1,2} \)) we have

\[
I(v_n) = I(v_n - \xi_j + \xi_j) = J_{\xi_j}(v_n - \xi_j) \to J_{\xi_j}(0) = I(\xi_j) = 0,
\]

a contradiction.

\[ \square \]

**Corollary 1.17.** If \( v \) is a non-constant solution to (HS) such that \( I(v) < \infty \), then

\[
\|v - v(\infty)\|_{L^\infty(\mathbb{R})} \geq \frac{\eta}{2}
\]

and

\[
\|v - v(-\infty)\|_{L^\infty(\mathbb{R})} \geq \frac{\eta}{2}
\]

**Proof.** If the statement is false, then there is a nonconstant solution \( v \) of (HS) such that \( I(v) < \infty \) and \( \|v - v(\infty)\|_{L^\infty} < \eta/2 \). By Lemma 1.3, we know that \( v(\infty) \in K(V) \). Without loss of generality, we may assume that \( v(\infty) = 0 \). The sequence \( u_n = v \) is a Palais-Smale sequence for \( I \), and \( I(u_n) = I(v) \) for all \( n \). We claim now that \( I(v) > 0 \), which would then contradict Lemma 1.16. Pick \( t_1 < t_2 \) such that \( |v(t_1)| = \|v\|_{L^\infty} > 0 \), and for all \( t \in (t_1, t_2) \), \( v(t) \in B_{\|v\|_{L^\infty}}(0) \). Then by Lemma 1.10 we have

\[
I(v) \geq \sqrt{2\alpha \left( \frac{\|v\|_{L^\infty}}{2} \right)} |v(t_2) - v(t_1)| > 0,
\]

which finishes the proof. \[ \square \]

**Corollary 1.18.** There is a \( \nu > 0 \) such that if \( I'(v) = 0 \) and \( v \) is non-constant, then \( I(v) > \nu \).
Proof. If there is no such \( \nu \), then there is a sequence of non-constant critical points \( v_n \) such that \( I(v_n) \to 0 \). By picking a subsequence, we may as well assume that \( v_n(-\infty) = 0 \) and \( v_n(\infty) = \xi_i \) for all \( n \). If \( v_n(-\infty) \neq v_n(\infty) \), then the \( v_n \) must travel between two equilibria, so by Lemma 1.10, \( I(v_n) \geq \frac{n}{2} \sqrt{2\alpha \left( \frac{4}{3} \right)} \), which contradicts \( I(v_n) \to 0 \). Thus, we must have \( v_n(\infty) = v_n(-\infty) = 0 \) for all \( n \). But then Lemma 1.16 implies that \( \|v_n\|_{L_\infty} \geq \frac{n}{2} \), so \( v_n \) must travel from \( \partial B_{n^4}(0) \) to \( \partial B_{n^2}(0) \). Thus, by Lemma 1.10, we must have \( I(v_n) \geq \frac{n}{4} \sqrt{2\alpha \left( \frac{4}{3} \right)} \), which is also impossible.

Now, we are ready to prove Proposition 1.15. Recall what we are trying to prove: For any fixed \( \rho \), there is an \( N(\rho) \) such that for \( n > N(\rho) \)

(i) \( \tau_{k_n} v_n(t) \in B_{2\rho}(0) \) for \( t < \xi(\rho) \)

(ii) \( \tau_{k_n} v_n(t) \in B_{2\rho}(\xi) \) for \( t > \bar{\xi}(\rho) \),

where \( v_n \) is a sequence with \( v_n(-\infty) = 0 \) and \( v_n(\infty) = \xi \) for all \( n \), and \( I(v_n) \to b > 0 \), \( f'(v_n) \to 0 \). (\( \xi(\rho), \bar{\xi}(\rho) \) are defined in (43).) For simplicity, we write \( v_n^1 \) for \( \tau_{k_n} v_n \). Thus, we have \( v_n^1(t) = v_n(t - k_n) \).

Proof. Notice that by Proposition 1.14 we have already shown that if \( \rho \) is small enough that \( \bar{\xi}(\rho) > \bar{\xi} + t^* + 1 \), then for all big enough \( n \), we have

(i') \( v_n^1(t) \in B_{\eta/2}(0) \) for \( t < \xi(\rho) \)

(ii') \( v_n^1(t) \in B_{\eta/2}(\xi) \) for \( t > \bar{\xi}(\rho) \)

Assume that (i) is false. We now want to get a contradiction. Since (i) is false, there must be a subsequence \( n_j \to \infty \) such that for every \( j \) there is a \( t_j < \xi(\rho) \) such that \( \tau_{k_{n_j}} v_{n_j}(t_j) \notin B_{2\rho}(0) \). We claim that \( t_j \to -\infty \) as \( j \to \infty \). If not, then we must have \( t_j \to \hat{\xi} \leq \xi(\rho) \) on a subsequence. Now, we claim that \( v(\hat{\xi}) = \lim_{j \to \infty} v_{n_j}^1(t_j) \). To see this:

\[
|v(\hat{\xi}) - v_{n_j}^1(t_j)| \leq |v(\hat{\xi}) - v_{n_j}^1(\hat{\xi})| + |v_{n_j}^1(\hat{\xi}) - v_{n_j}^1(t_j)|
\]  

The first term on the right in (73) \( \to 0 \) as \( j \to \infty \) because of pointwise convergence (indeed, we have \( L_{loc}^\infty \) convergence of \( v_n^1 \) to \( v \)). The second term on the right in (73) also \( \to 0 \) because \( \{\tau_{k_n} v_n\} \) is equicontinuous. But this implies that \( |v(\hat{\xi})| \geq 2\rho \), which contradicts the definition of \( \xi(\rho) \) from (43). Now, let

\[
q_j(t) := \begin{cases} 
  v_{n_j}^1(t) & \text{for } t < \xi(\rho) \\
  (\xi(\rho) + 1 - t)(v_{n_j}^1(\xi(\rho))) & \text{for } \xi(\rho) \leq t \leq \xi(\rho) + 1 \\
  0 & \text{for } t > \xi(\rho)
\end{cases}
\]
We have $\|q_j\|_{L^\infty} < \eta/2$, since for $t < t(\rho)$, $q_j(t) = v^1_{n_j}(t) \in B_{\eta/2}(0)$, while for $t$ between $t(\rho)$ and $t(\rho) + 1$, $q_j(t)$ is a convex combination of elements of $B_{\eta/2}(0)$. Notice that we have $q_j \in W^{1,2}$, and $\|q_j\|_{W^{1,2}}$ is bounded independently of $j$ and $\rho$. (The proof of this is similar to showing the boundedness of $\psi_n$ in Proposition 1.13, where we use Proposition 1.14.) Pick $k_j \in \mathbb{Z}$ such that $t_j + k_j \in [0, 1)$, and consider the sequence $q^1_j := q(t-k_j)$. Notice that this means that $q^1_j(t + k_j) = q_j(t_j) = v^1_{n_j}(t_j) \notin B_{2\rho}(0)$. But we also have $\|q_j\|_{W^{1,2}} = \|q^1_j\|_{W^{1,2}}$, so (along a subsequence) $q^1_j \rightharpoonup q$ in $W^{1,2}$. Now, we also have $t_j + k_j \to \hat{t} \in [0, 1]$, and $q^1_j \to q$ in $L^\infty_{loc}$, so we have

$$|q(\hat{t}) - q^1_j(t_j + k_j)| \leq |q(\hat{t}) - q^1_j(\hat{t})| + |q^1_j(t) - q^1_j(t_j + k_j)|$$

(74)

As in (73), the right side in (74) → 0 as $j \to \infty$, hence $|q(\hat{t})| \geq 2\rho$, so $q$ is not a constant. Because of the $L^\infty$ bounds on $q_j$, we must also have $\|q\|_{L^\infty} \leq \eta/2$. Moreover, by the weak lower semi-continuity of $I$, we have $I(q) < \infty$. Thus, if we can show that $q$ satisfies (HS), we'll have a contradiction to Corollary 1.17. It suffices to show

$$\int_{\mathbb{R}} (\langle \dot{q}, \dot{\varphi} \rangle - \langle V_q(t, q), \varphi \rangle) dt = 0$$

for all $\varphi \in C_c^\infty(\mathbb{R})$. So, suppose that $\varphi \in C_c^\infty(\mathbb{R})$, and that $supp(\varphi) \subset (a, c)$, and pick $j$ so large that $c - k_j < \hat{t}(\rho)$. (That this is possible relies on the fact that $k_j \to \infty$, which follows from the fact that $t_j \to -\infty$.) Then, changing variables, using the 1-periodicity of $V$ and noting that because $c - k_j < \hat{t}(\rho)$, we have $q_j(t) = v^1_{n_j}(t)$ for all $t \in (a - k_j, c - k_j)$, we have

$$\int_{\mathbb{R}} \langle \dot{q}^1_j, \dot{\varphi} \rangle - \langle V_q(t, q^1_j), \varphi \rangle dt = \int_{a}^{c} \langle \dot{q}_j(t - k_j), \dot{\varphi}(t) \rangle - \langle V_q(t, q_j(t - k_j)), \varphi(t) \rangle dt$$

$$= \int_{a- k_j}^{c-k_j} \langle \dot{q}_j(t), \dot{\varphi}(t + k_j) \rangle - \langle V_q(t + k_j, q_j(t)), \varphi(t + k_j) \rangle dt$$

$$= \int_{a- k_j}^{c-k_j} \langle \dot{q}_j(t), \dot{\varphi}(t + k_j) \rangle - \langle V_q(t, q_j(t)), \varphi(t + k_j) \rangle dt$$

$$= \int_{\mathbb{R}} \langle \dot{v}^1_{n_j}, \dot{\varphi}_j(t) \rangle - \langle V_q(t, v^1_{n_j}(t)), \varphi_j(t) \rangle dt$$

$$= I'(v^1_{n_j}) \varphi_j,$$

where $\varphi_j(t) := \varphi(t + k_j)$. Now $\|\varphi_j\|_{W^{1,2}} = \|\varphi\|_{W^{1,2}}$, so we have $I'(v^1_{n_j})\varphi_j \to 0$ as $j \to \infty$. Now, by the weak convergence of $q^1_j$ to $q$, we have

$$\int_{\mathbb{R}} \langle \dot{q}^1_j, \dot{\varphi} \rangle dt \to \int_{\mathbb{R}} \langle \dot{q}, \dot{\varphi} \rangle dt$$

(76)
as $j \to \infty$, and because of the $L^\infty_{loc}$ convergence we have
\[
\int_{\mathbb{R}} \langle V_q(t, q_j^1), \varphi \rangle dt \to \int_{\mathbb{R}} \langle V_q(t, q), \varphi \rangle dt \tag{77}
\]
as $j \to \infty$ because $\varphi$ has compact support. Combining (76) and (77), we have
\[
\int_{\mathbb{R}} \langle \dot{q}, \dot{\varphi} \rangle - \langle V_q(t, q), \varphi \rangle dt = \lim_{j \to \infty} I'(v_n^1)\varphi = 0
\]
This then contradicts Corollary 1.17, which proves (i). The proof of (ii) is similar.

Now, suppose we have a sequence $v_n$ with $I(v_n) \to b > 0$ and $I'(v_n) \to 0$. We know what
to do if there are only two intervals on which $v_n$ spends an unbounded amount of time in $B_{n/2}(K(V))$. We now seek to reduce the general problem to exactly this case. To do this we
need the following lemma:

**Lemma 1.19.** Suppose that $I(v_n) \to b > 0$ and $I'(v_n) \to 0$. Moreover, suppose that
\((t_{n_j}, \bar{t}_{n_j})\)
is an interval whose length $\to \infty$ as $n \to \infty$, and $v_n(t) \in B_{n/2}(\xi)$ for some fixed $\xi \in K(V)$,
all $t \in (t_{n_j}, \bar{t}_{n_j})$ and all $n$. Let $t_{n_j}^j$ be any point in $(t_{n_j}, \bar{t}_{n_j})$ such that
\begin{enumerate}[(a)]
  \item $t_{n_j}^j - t_{n_j} \to \infty$ as $n \to \infty$ and
  \item $\bar{t}_{n_j} - t_{n_j}^j \to \infty$ as $n \to \infty$.
\end{enumerate}
Then \(\|v_n - \xi\|_{L^\infty_{[-1, 1]}} \to 0\).

**Proof.** Pick $k_n \in \mathbb{Z}$ such that $t_{n_j}^j - k_n \in [0, 1)$, and consider the sequence $\tilde{v}_n(t) := v_n(t + k_n)$.
Then we have $I(\tilde{v}_n) = I(v_n)$, so $I(\tilde{v}_n)$ is bounded. Because of the normalization provided by
the choice of the $k_n$, this implies that (on a subsequence) $\tilde{v}_n \to v$ in $\hat{E}$. We claim now that
\begin{enumerate}[(i)]
  \item $v$ satisfies (HS)
  \item $\|v - \xi\|_{L^\infty} \leq \frac{\eta}{2}$
\end{enumerate}
Assuming these, then Corollary 1.17 implies that $v \equiv \xi$. But weak convergence in $\hat{E}$ implies
local uniform convergence, so
\[
\|\tilde{v}_n - \xi\|_{L^\infty_{[-1, 2]}} \to 0.
\tag{78}
\]
Now, if $t_{n_j}^j - 1 \leq t \leq t_{n_j}^j + 1$, then $t_{n_j}^j - 1 - k_n \leq t - k_n \leq t_{n_j}^j + 1 - k_n$ hence $t - k_n \in [-1, 2]$.
Therefore
\[
\|\tilde{v}_n - \xi\|_{L^\infty_{[-1, 2]}} \geq \|\tilde{v}_n - \xi\|_{L^\infty_{[t_{n_j}^j - k_n - 1, t_{n_j}^j - k_n + 1]}}
= \|v_n - \xi\|_{L^\infty_{[t_{n_j}^j - 1, t_{n_j}^j + 1]}}.
\tag{79}
\]
which is what we want.

To see (i), pick \( \varphi \in C_c^\infty(\mathbb{R}) \), and suppose that \( \text{supp}(\varphi) \subset (a,c) \). Taking \( \varphi_n(t) := \varphi(t+k_n) \), we have

\[
\int_\mathbb{R} \langle \dot{v}_n, \varphi \rangle - \langle V_q(t, \bar{v}_n), \varphi \rangle dt = \int_a^c \langle \dot{v}_n(t-k_n), \varphi(t) \rangle - \langle V_q(t, v_n(t-k_n)), \varphi(t) \rangle dt \\
= \int_{a-k_n}^{c-k_n} \langle \dot{v}_n(t), \varphi(t+k_n) \rangle - \langle V_q(t+k_n, v_n(t)), \varphi(t+k_n) \rangle dt \\
= \int_{a-k_n}^{c-k_n} \langle \dot{v}_n(t), \varphi(t+k_n) \rangle - \langle V_q(t, v_n(t)), \varphi(t+k_n) \rangle dt \\
= I'(v_n) \varphi_n \to 0
\]

since \( \varphi_n \) is bounded in \( W^{1,2} \).

But

\[
\int_\mathbb{R} \langle \dot{v}_n, \varphi \rangle dt \to \int_\mathbb{R} \langle \dot{v}, \varphi \rangle dt
\]

by the weak convergence in \( \dot{\mathcal{E}} \). Because \( \varphi \) has compact support, we also have

\[
\int_\mathbb{R} \langle V_q(t, \bar{v}_n), \varphi \rangle dt \to \int_\mathbb{R} \langle V_q(t, v), \varphi \rangle dt.
\]

Thus, we must have

\[
\int_\mathbb{R} \langle \dot{v}, \varphi \rangle - \langle V_q(t, v), \varphi \rangle dt = 0
\]

for all \( \varphi \in C_c^\infty \), hence \( v \) satisfies (HS).

It remains only to prove that (ii) is satisfied. Note that \( v_n(t) \in B_{\eta/2}(\xi) \) for \( t \in (\bar{t}_{n_j}, \bar{t}_{n_j}) \).

Since \( v \) is the pointwise limit of the \( \bar{v}_n \), in order to show (ii), it suffices to show that for a fixed \( \hat{t} \), we have \( \bar{v}_n(\hat{t}) \in B_{\eta/2}(\xi) \) for all large \( n \). (How large \( n \) must be might depend on \( \hat{t} \).) Because \( \bar{v}_n(\hat{t}) = v_n(\hat{t} + k_n) \), it suffices to show that \( \hat{t} + k_n \in (\bar{t}_{n_j}, \bar{t}_{n_j}) \) for all large \( n \). Equivalently, we need to show that \( \hat{t} \in (\bar{t}_{n_j} - k_n, \bar{t}_{n_j} - k_n) \) for all large \( n \). We claim that we have

\[
\bar{t}_{n_j} - k_n \to -\infty \quad \text{and} \quad \bar{t}_{n_j} - k_n \to \infty,
\]

which would then imply our claim. Notice that

\[
\bar{t}_{n_j} - k_n = (\bar{t}_{n_j} - t_{n_j}^i) - (t_{n_j}^i - k_n).
\]

But the second term on the right in (82) is bounded by the definition of \( k_n \), and by (a) of our assumptions, \( (\bar{t}_{n_j} - t_{n_j}^i) \to -\infty \) as \( n \to \infty \). Thus (82) implies that \( \bar{t}_{n_j} - k_n \to -\infty \) as \( n \to \infty \). Similarly,

\[
\bar{t}_{n_j} - k_n = (\bar{t}_{n_j} - t_{n_j}^i) - (t_{n_j}^i - k_n).
\]
The second term on the right in (83) is bounded by the definition of $k_n$ while the first tends to $\infty$ as $n \to \infty$ by assumption (b). Therefore, (83) implies that $t_{n_j} - k_n \to \infty$ as $n \to \infty$, and (81) is proven.

With these preliminaries out of the way, we can finally turn to the general case when there are more than two intervals where $v_n \in B_{\eta/2}(K(V))$. We know what to do if there are only two intervals on which $v_n$ spends an increasing amount of time in $B_{\eta/2}(K(V))$. Suppose that the only intervals whose lengths $\to \infty$ as $n \to \infty$ and in which $v_n(t) \in B_{\eta/2}(K(V))$ are

$$(-\infty, \bar{t}_{n_1}), (\bar{t}_{n_2}, \bar{t}_{n_2}), (\bar{t}_{n_3}, \bar{t}_{n_3}), \ldots, (\bar{t}_{l(n)}, \infty).$$

By passing to a subsequence if necessary, we may assume that $l(n)$ is independent of $n$. Call this constant value $l$. We must have $l \in \mathbb{N}$ and $l \geq 2$. Since we have already dealt with the case of $l = 2$, suppose $l > 2$. Let us suppose now that

1. $v_n(-\infty) = 0$ for all $n$
2. $v_n(\infty) = \xi_l$ for all $n$

By passing to a subsequence, we may also assume that

3. $v_n(t) \in B_{\eta/2}(\xi_2)$ for all $t \in (t_{n_2}, \bar{t}_{n_2})$ and all $n$. 
4. $v_n(t) \in B_{\eta/2}(\xi_3)$ for all $t \in (t_{n_3}, \bar{t}_{n_3})$

and in general

$$(j) \ v_n(t) \in B_{\eta/2}(\xi_j) \text{ for all } t \in (t_{n_j}, \bar{t}_{n_j})$$

for $j \leq l$.

The content of the next theorem is that we may split the sequence $v_n$ into $l - 1$ Palais-Smale sequences such that on each one, there are only two intervals where an unbounded amount of time is spent in $B_{\eta/2}(K(V))$.

**Proposition 1.20.** Let $v_n$ be as above. Then there are PS sequences $v_i^n$ for $i = 1, 2, \ldots, l - 1$ such that

1. $v_i^n(t) \in B_{\eta/2}(\xi_i)$ for $t < \bar{t}_{n_i}$
2. $v_i^n(t) \in B_{\eta/2}(\xi_{i+1})$ for $t > t_{n_{i+1}}$
(iii) \((-\infty, \bar{t}_n)\) and \((\bar{t}_{n+1}, \infty)\) are the only intervals with unbounded length where \(v^i_n(t) \in B_{\eta/2}(K(V))\).

(iv) \(|I(v^i_n) - \int_{t_n}^{t_{n+1}} (1/2|\dot{v}_n|^2 - V(t, v_n)) dt| \to 0\) as \(n \to \infty\) for any \(i = 1, 2, \ldots, l - 1\).

Proof. For \(i = 2, 3, \ldots, l - 2\), pick points \(t^i_n \in (\bar{t}_n, \bar{t}_n)\) such that assumptions (a) and (b) of Lemma 1.19 hold. (For example, we could take \(t^i_n\) to be the midpoint of \((\bar{t}_n, \bar{t}_n)\).) Then, let

\[
v^i_n(t) := \begin{cases} 
  \xi_i & \text{for } t < t^i_n - 1 \\
  (t - t^i_n)(\xi_i - v_n(t^i_n)) + \xi_i & \text{for } t^i_n - 1 \leq t \leq t^i_n \\
  v_n(t) & \text{for } t^i_n < t < t^{i+1}_n \\
  (t^{i+1}_n + 1 - t)(v_n(t^{i+1}_n) - \xi_{i+1}) - \xi_{i+1} & \text{for } t^{i+1}_n \leq t \leq t^{i+1}_n + 1 \\
  \xi_{i+1} & \text{for } t > t^{i+1}_n + 1
\end{cases}
\]

For \(i = 1\), we have \(t^1_n = -\infty\) and we take \(v^1_n(t) = v_n(t)\) for \(t < t^2_n\). For \(t^2_n \leq t \leq t^2_n + 1\), we interpolate as above between \(v_n(t^2_n)\) and \(\xi_2\), and then take \(v^1_n(t) = \xi_2\) for all \(t > t^2_n + 1\). For \(i = l - 1\), we have \(t^i_n = \infty\), and so we take \(v^{l-1}_n(t) = v_n(t)\) for \(t > t^{l-1}_n\). For \(t^{l-1}_n \leq t \leq t^{l-1}_n\), we interpolate between \(\xi_{l-1}\) and \(v_n(t^{l-1}_n)\), while for \(t < t^{l-1}_n\), \(v^{l-1}_n(t) = \xi_{l-1}\). In any case, the arguments that follow are the same, so we ignore these two special cases. Notice that we have obtained \(v^i_n\) from \(v_n\) by linear interpolation at \(t^i_n, t^{i+1}_n\) between \(v_n\) and \(\xi_i, \xi_{i+1}\), respectively. In addition, for each \(n\), the chain \(\{v^1_n, v^2_n, \ldots, v^{l-1}_n\}\) connects 0 to \(\xi_l\), and “shadows” \(v_n\) in the sense that \(v^i_n\) is \(W^{1,2}\) close to \(v_n\) for long intervals.

For future reference, note that if \(\|v_n\|_{L^\infty(\mathbb{R})} \leq C\), then we must have \(\|v^i_n\|_{L^\infty(\mathbb{R})} \leq C + \eta\).

To see this, note that for \(t \in (t^i_n, t^{i+1}_n)\), we have \(v^i_n(t) = v_n(t)\), hence \(\|v^i_n\|_{L^\infty(t^i_n, t^{i+1}_n)} \leq C\). If \(t \leq t^i_n\), then \(v^i_n(t) \in B_{\eta/2}(\xi_i)\). But

\[
|v^i_n(t)| \leq |v^i_n(t) - v_n(t^i_n)| + |v_n(t^i_n)| \leq |v^i_n(t) - \xi_i| + |\xi_i - v_n(t^i_n)| + \|v_n\|_{L^\infty} \leq \frac{\eta}{2} + \frac{\eta}{2} + C = C + \eta.
\]

The case for \(t \geq t^{i+1}_n\) is similar. We now verify (i)-(iv).

(i): Suppose that \(t < \bar{t}_n\). If \(t \leq t^i_n\), then (i) is obvious, since we have \(v^i_n(t) = \xi_i\) or \(v^i_n(t) = \xi_{i+1}\) is a convex combination of elements in \(B_{\eta/2}(\xi_i)\). If \(t^i_n < t < \bar{t}_n\), then \(v^i_n(t) = v_n(t)\), and on this interval \(v_n(t) \in B_{\eta/2}(\xi_i)\).

(ii): Suppose that \(t > \bar{t}_{n+1}\). If \(t \geq t^{i+1}_n\), then \(v^i_n(t) = \xi_{i+1}\) or \(v^i_n(t) = \xi_{i+1}\) is a convex combination of elements in \(B_{\eta/2}(\xi_{i+1})\). If \(t^{n+1}_n < t < t^{i+1}_n\), then \(v^i_n(t) = v_n(t)\) and on this interval \(v_n(t) \in B_{\eta/2}(\xi_{i+1})\).
(iii): Suppose that there is another interval \((t_n^a, t_n^c)\) such that \(t_n^c - t_n^a \to \infty\) as \(n \to \infty\) and on which \(v_n(t) \in B_{n/a}(K(V))\). Then \((t_n^a, t_n^b) \subset (\tilde{t}_{n+1}, \tilde{t}_{n+1})\). But by Lemma 1.12, \(M \geq \tilde{t}_{n+1} - \tilde{t}_{n+1} \geq t_n^b - t_n^a \to \infty\), which is impossible. Thus there can be no such interval.

(iv): Notice that we have

\[
0 \leq I(v_n^i) - \int_{t_n^a}^{t_{n+1}^a} \left( \frac{1}{2} |\dot{v}_n^i|^2 - V(t, v_n^i) \right) dt
\]

\[
= \int_{t_n^a}^{t_{n+1}^a} \left( \frac{1}{2} |\dot{v}_n^i|^2 - V(t, v_n^i) \right) dt + \int_{t_n^a}^{t_{n+1}^a} \left( \frac{1}{2} |\dot{v}_n^i|^2 - V(t, v_n^i) \right) dt
\]

\[
= \int_{t_n^a}^{t_{n+1}^a} \left( \frac{1}{2} |\xi_i - v_n(t_n^i)|^2 - V(t, \text{linear in } t \text{ piece}) \right) dt
\]

\[
+ \int_{t_n^a}^{t_{n+1}^a} \left( \frac{1}{2} |\xi_{i+1} - v_n(t_{n+1}^i)|^2 - V(t, \text{linear in } t \text{ piece}) \right) dt
\]

\[
= \frac{1}{2} \xi_i - v_n(t_n^i)^2 + \frac{1}{2} \xi_{i+1} - v_n(t_{n+1}^i)^2 + \int_{t_n^a}^{t_{n+1}^a} -V(t, \text{linear in } t \text{ piece}) dt
\]

\[
+ \int_{t_n^a}^{t_{n+1}^a} -V(t, \text{linear in } t \text{ piece}) dt.
\]

Since \(v_n(t_n^i) \to \xi_i\) and \(v_n(t_{n+1}^i) \to \xi_{i+1}\) by Lemma 1.19, all the terms on the right in (87) go to 0 as \(n \to \infty\).

It remains only to see that the sequences \(v_n^i\) are PS sequences. Since \(I(v_n^i)\) is bounded, it remains only to see that \(I'(v_n^i) \to 0\) as \(n \to \infty\) for \(i = 1, 2, \ldots, l - 1\). Let \(\varphi \in W^{1,2}\), and suppose that \(\|\varphi\|_{W^{1,2}} \leq 1\). We want to show that \(\|I'(v_n^i)\varphi\| \to 0\) independently of such \(\varphi\). We have

\[
I'(v_n^i)\varphi = \int_{t_n^a}^{t_{n+1}^a} \left( \langle \xi_i - v_n(t_n^i), \varphi \rangle - V_q(t, \text{linear in } t \text{ piece}), \varphi \rangle \right) dt
\]

\[
+ \int_{t_n^a}^{t_{n+1}^a} \langle \dot{v}_n, \varphi \rangle - V_q(t, v_n, \varphi) \right) dt
\]

\[
+ \int_{t_n^a}^{t_{n+1}^a} \langle \xi_{i+1} - v_n(t_{n+1}^i), \varphi \rangle - V_q(t, \text{linear in } t \text{ piece}), \varphi \rangle dt.
\]

The second two terms in (88) \(\to 0\) as \(n \to \infty\) independently of \(\varphi\) because:

\[
\left| \int_{t_n^a}^{t_{n+1}^a} \left( \langle \xi_i - v_n(t_n^i), \varphi \rangle \right) dt \right| \leq |\xi_i - v_n(t_n^i)| \|\varphi\|_{L^2(t_n^a, t_{n+1}^a)}
\]

\[
\leq |\xi_i - v_n(t_n^i)|
\]
and
\[
\int_{t_{i-1}^n}^{t_i} \langle V_q(t, \text{linear in } t \text{ piece}), \varphi \rangle dt \leq \|V_q(t, \text{linear in } t \text{ piece})\|_{L^\infty(t_{i-1}^n, t_i^+)} \|\varphi\|_{L^2} \quad (90)
\]
\[
\leq \|V_q(t, \text{linear in } t \text{ piece})\|_{L^\infty(t_{i-1}^n, t_i^+)}.
\]

Both (89) and (90) go to 0 as \( n \to \infty \), which follows from the uniform convergence of the linear piece to 0. This in turn follows from Lemma 1.19.

To finish the proof, it remains to verify that
\[
\int_{t_n^i}^{t_n^{i+1}} \langle \dot{v}_n, \dot{\varphi} \rangle - \langle V_q(t, v_n), \varphi \rangle \, dt \to 0 \quad (91)
\]
as \( n \to \infty \), independently of \( \varphi \).

For any \( \varphi \in W^{1,2} \), we define \( \varphi_n \) by:
\[
\varphi_n(t) := \begin{cases} 
0 & \text{for } t < t_{i-1}^n \\
(t - t_{i-1}^n + 1)\varphi(t_{i-1}^n) & \text{for } t_{i-1}^n - 1 \leq t \leq t_i^i \\
\varphi(t) & \text{for } t_{i-1}^n < t < t_{i+1}^i \\
(t_{i+1}^i + 1 - t)\varphi(t_{i+1}^i) & \text{for } t_{i+1}^i \leq t \leq t_{i+1}^i + 1 \\
0 & \text{for } t > t_{i+1}^i 
\end{cases}
\]

Thus, \( \varphi_n \) is obtained from \( \varphi \) by linear interpolation between \( \varphi \) and 0 at \( t_{i-1}^n \) and \( t_{i+1}^i \). We claim that \( \varphi_n \) is bounded in \( W^{1,2} \) independently of \( n \) and \( \varphi \). To see this, we have: (using the fact that \( \|\varphi\|_{L^\infty(\mathbb{R})} \leq C\|\varphi\|_{W^{1,2}(\mathbb{R})} \), see [1])
\[
\|\varphi_n\|_{L^2}^2 \leq \|\varphi\|_{L^2}^2 + 2\|\varphi\|_{L^\infty}^2 \\
\leq (2C^2 + 1)\|\varphi\|_{W^{1,2}(\mathbb{R})}^2 \leq 2C^2 + 1
\]
and
\[
\|\dot{\varphi}_n\|_{L^2}^2 \leq \|\dot{\varphi}\|_{L^2}^2 + 2\|\varphi\|_{L^\infty}^2 \\
\leq (2C^2 + 1)\|\varphi\|_{W^{1,2}(\mathbb{R})}^2 \leq 2C^2 + 1
\]
Therefore, $|I'(v_n)\varphi_n| \leq K\|I'(v_n)\| \to 0$ as $n \to \infty$ uniformly for $\|\varphi\|_{W^{1,2}} \leq 1$. We have

$$I'(v_n^i)\varphi - I'(v_n)\varphi_n = \int_{t^i_{n-1}}^{t^i_n} \langle \xi_i - v_n(t^i_n), \varphi \rangle - \langle V_q(t, \text{linear in } t \text{ piece}), \varphi \rangle \, dt$$

$$+ \int_{t^i_{n-1}}^{t^i_{n+1}} \langle \xi_i+1 - v_n(t^i_{n+1}), \varphi \rangle - \langle V_q(t, \text{linear in } t \text{ piece}), \varphi \rangle \, dt$$

$$- \int_{t^i_{n-1}}^{t^i_n} \langle \tilde{v}_n, \varphi(t^i_n) \rangle - \langle V_q(t, v_n), (t - t^i_n + 1)\varphi(t^i_n) \rangle \, dt$$

$$- \int_{t^i_{n-1}}^{t^i_{n+1}} \langle \tilde{v}_n, \varphi(t^i_{n+1}) \rangle - \langle V_q(t, v_n), (t^i_{n+1} + 1 - t)\varphi(t^i_{n+1}) \rangle \, dt$$

(92)

We already know that the first two terms in (92) go to 0 as $n \to \infty$ independently of $\varphi$, so we turn our attention to showing this for the second two. This will be a consequence of Lemma 1.19, which implies $\|v_n - \xi\|_{L^\infty[t^i_{n-1},t^i_n]} \to 0$, hence $V_q(t,v_n(t)) \to 0$ uniformly on $[t^i_{n-1},t^i_n]$ as $n \to \infty$. Therefore,

$$\left| \int_{t^i_{n-1}}^{t^i_n} \langle V_q(t,v_n), (t - t^i_n + 1)\varphi(t^i_n) \rangle \, dt \right| \leq \|\varphi\|_{L^\infty} \|V_q(t,v_n)\|_{L^\infty[t^i_{n-1},t^i_n]}$$

$$\leq C\|V_q(t,v_n)\|_{L^\infty[t^i_{n-1},t^i_n]} \to 0$$

as $n \to \infty$. The

$$\int_{t^i_{n-1}}^{t^i_{n+1}} \langle V_q(t,v_n), (t^i_{n+1} + 1 - t)\varphi(t^i_{n+1}) \rangle \, dt$$

term in (92) is dealt with similarly. Next, we turn our attention to the

$$\int_{t^i_{n-1}}^{t^i_n} \langle \tilde{v}_n, \varphi(t^i_n) \rangle \, dt$$

term in (92) We have the following:

$$\left| \int_{t^i_{n-1}}^{t^i_n} \langle \tilde{v}_n, \varphi(t^i_n) \rangle \, dt \right| = \sum_{k=1}^{d} \left| \varphi^k(t^i_n) \int_{t^i_{n-1}}^{t^i_n} \dot{v}_n^k(t) \, dt \right|$$

(94)

where $\varphi^k(t), v_n^k(t)$ are the components of $v(t), \varphi(t) \in \mathbb{R}^d$ for $k = 1, 2, \ldots, d$. Therefore, (94) implies that

$$\left| \int_{t^i_{n-1}}^{t^i_n} \langle \tilde{v}_n, \varphi(t^i_n) \rangle \, dt \right| = \sum_{k=1}^{d} \left| \varphi^k(t^i_n) \left( v_n^k(t^i_n) - v_n^k(t^i_n - 1) \right) \right|$$

$$\leq \|\varphi\|_{L^\infty} \sum_{k=1}^{d} \left| (v_n^k(t^i_n) - v_n^k(t^i_n - 1)) \right|$$

$$\leq C \sum_{k=1}^{d} \left| (v_n^k(t^i_n) - v_n^k(t^i_n - 1)) \right| \to 0$$

(95)
because of Lemma 1.19. We deal with the $\int_{t_n}^{t_{n+1}} \langle \dot{v}_n, \varphi \rangle dt$ term similarly.

Altogether, we have

$$I'(v_n^i) \varphi = \text{a sum of terms that } \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ independent of } \varphi$$

and so $I'(v_n^i) \rightarrow 0$ as $n \rightarrow \infty$, which concludes the proof. \qed

Suppose then that we have a sequence $v_n \in \hat{E}$ such that $I(v_n) \rightarrow b > 0$ and $I'(v_n) \rightarrow 0$. Moreover, suppose that $v_n(-\infty) = 0$ and $v_n(\infty) = \xi \neq 0$ for all $n$. By Lemma 1.3, we know that $v_n$ is bounded in $L^\infty(\mathbb{R})$. We also know that there is a finite number of intervals, say $l$, whose lengths are unbounded in $n$ and on which $v_n(t) \in B_{\eta/2}(\xi_j)$ for some $\xi_j \in K(V)$, $j = 1, 2, \ldots, l$. By Proposition 1.20, there are $l - 1$ PS sequences $v_n^i \in \hat{E}$ such that each one has only two intervals whose lengths are unbounded in $n$ and on which $v_n^i(t) \in B_{\eta/2}(K(V))$. Moreover, we can pick a subsequence of $v_n$ such that the $v_n^i$ form a chain from 0 to $\xi$.

Notice that by (iv) of Proposition 1.20 and the $L^\infty$ boundedness of $v_n$, we know $I(v_n^i)$ is bounded. Passing to subsequences yet again, we may assume there exist $b_i > 0$ such that for $i = 1, 2, \ldots, l - 1$, $I(v_n^i) \rightarrow b_i$. By Proposition 1.13, for each sequence $v_n^i$, there is a sequence $k_n^i \in \mathbb{Z}$ such that $\|\tau_{k_n^i} v_n^i - v^i\|_{W^{1,2}} \rightarrow 0$, where $v^i$ is a critical point of $I$ such that $I(v^i) = b_i$.

How is $\sum_{i=1}^{l-1} b_i$ related to $b$?

We claim $\sum_{i=1}^{l-1} b_i = b$. To see this, note that (since $t_1 = -\infty$ and $t_l = +\infty$) we have

$$I(v_n) = \sum_{i=1}^{l-1} \int_{t_i}^{t_{i+1}} \left( \frac{1}{2} |\dot{v}_n|^2 - V(t, v_n) \right) dt. \quad (96)$$

By (iv) of Proposition 1.20

$$I(v_n^i) - \int_{t_i}^{t_{i+1}} \left( \frac{1}{2} |\dot{v}_n|^2 - V(t, v_n) \right) dt \rightarrow 0 \quad (97)$$

as $n \rightarrow \infty$ for $i = 1, 2, \ldots, l - 1$. Thus, combining (96) and (97), we have

$$\left| b - \sum_{i=1}^{l-1} b_i \right| \leq |b - I(v_n)| + \left| I(v_n) - \sum_{i=1}^{l-1} I(v_n^i) \right| + \left| \sum_{i=1}^{l-1} (I(v_n^i) - b_i) \right|$$

$$= |b - I(v_n)| + \sum_{i=1}^{l-1} \left( \left( \int_{t_i}^{t_{i+1}} \frac{1}{2} |\dot{v}_n|^2 - V(t, v_n) dt \right) - I(v_n^i) \right)$$

$$+ \left| \sum_{i=1}^{l-1} (I(v_n^i) - b_i) \right|. \quad (98)$$

Letting $n \rightarrow \infty$ then implies that $\sum_{i=1}^{l-1} b_i = b$.

Altogether, we have proven the following result on the structure of (PS) sequences:
Theorem 1.21. If \( v_n \in \hat{E} \) is a sequence such that

\begin{enumerate}[\text{(C1)}]
    \item \( v_n(-\infty) = 0 \) and \( v_n(\infty) = \tilde{\xi} \) for some fixed \( \tilde{\xi} \in K(V) \) and all \( n \)
    \item \( I(v_n) \to b > 0 \) and \( I'(v_n) \to 0 \).
\end{enumerate}

Then, there exists a finite set of solutions to (HS) \( v^1, v^2, \ldots, v^m \) such

\begin{enumerate}[(i)]
    \item \( \sum_{i=1}^{m} I(v^i) = b \)
    \item \( v^1(-\infty) = 0, \ v^m(\infty) = \tilde{\xi} \), and \( v^i(\infty) = v^{i+1}(-\infty) \in K(V) \) for \( i = 1, 2, \ldots, m - 1 \).
\end{enumerate}

Here \( m + 1 \) is the number of intervals with unbounded length in \( n \) on which \( v_n(t) \in B_{\eta/2}(K(V)) \).

Remark: (1) If we have a sequence \( v_n \) such that \( I(v_n) \to b > 0 \) and \( I'(v_n) \to 0 \), then by Lemma 1.3, we know that \( \|v_n\|_{L^\infty(\mathbb{R})} \leq C \). But then \( \|v^i_n\|_{L^\infty} \leq C + \eta \). By the \( W^{1,2} \) convergence of \( \tau_{k_h} v^i_n \) to \( v^i \) from Proposition 1.13, we know that in fact \( \tau_{k_h} v^i_n \) converges to \( v^i \) in \( L^\infty \), hence \( \|v^i_n\|_{L^\infty} \leq C + \eta \).

(2) It is possible to give a precise convergence statement about the manner in which \( v_n \) converges to a “chain” of solutions of (HS). Such a statement would be very similar to Proposition 3.10 in [3]. However, we find it more convenient to use Proposition 1.20 to break \( v_n \) into \( m \) (PS) sequences \( v^i_n \), and then apply Proposition 1.13 to each \( v^i_n \).

Observe also that we have not excluded the case of homoclinic solutions of (HS) in Theorem 1.21.

2 Mountain Pass Points

Finally, we can prove the existence of solutions of (HS) that are not minimizers. Let us consider the case when \( K(V) = \{0, \xi\} \). It is known that there exist heteroclinic connections between the two equilibria by minimizing \( I \) over an appropriate subset of \( \hat{E} \) (see [3], [5]). Suppose that \( q \) is a heteroclinic going from 0 to \( \xi \). Notice that by (V1), \( \bar{q}(t) := q(t + 1) \) is also a minimizer, and \( I(\bar{q}) = I(q) \). Once we have two minimizers, we can ask ourselves if there is a critical point of mountain pass type “between” them. To do this, we make the following assumption about \( I \) and \( q \).

\( q \) is an isolated minimizer of \( I \)
For example, the assumptions about the potential $V$ imply that $I$ is actually $C^2$, so if $I''(q)(\varphi, \varphi) \geq a\|\varphi\|^2$ for some $a > 0$, then the assumption above would be satisfied.

Let $\tilde{J}(u) := J_q(u) - I(q)$. Notice that by Proposition 1.6, $\tilde{J} \in C^1(E, \mathbb{R})$ and $\tilde{J}(u) \geq 0$ for all $u \in E$. Let

$$c := \inf_{h \in \Gamma} \max_{0 \leq s \leq 1} \tilde{J}(h_s)$$

where $\Gamma := \{h \in C([0, 1], E) \mid h(0) \equiv 0 \text{ and } h(1)(t) = q(t + 1) - q(t)\}$. Notice that by our assumption about $q$ being an isolated local minimizer, $c > 0$. In fact, we have the following proposition:

**Proposition 2.1.** If $c = 0$, then for every $r \in (0, \|q(t+1) - q(t)\|_{W^{1,2}})$, there is a $k = k(r) \in \mathbb{Z}$ and a solution $\tilde{q}$ of (HS) such that $I(\tilde{q}) = I(q)$ and $\|\tilde{q} - \tau_kq\|_{W^{1,2}} = r$.

**Proof.** If $c = 0$, there is a sequence $h_n \in \Gamma$ such that $\max_{0 \leq s \leq 1} \tilde{J}(h(s)) < 1/n$. For any $r \in (0, \|q(t+1) - q(t)\|_{W^{1,2}})$, let $s_{n,r}$ be the smallest $s$ for which $\|h_n(s)\|_{W^{1,2}} = r$. Then $\tilde{J}(h_n(s_{n,r})) < 1/n$, so $h_n(s_{n,r})$ is a minimizing sequence for $\tilde{J}$. We next apply Ekeland’s principle: for any $n$ and $u \in E$ with $\tilde{J}(u) < \inf_{x \in E} \tilde{J}(x) + 1/n$, there is a $v \in E$ such that

(a) $\tilde{J}(v) \leq \tilde{J}(u)$

(b) $\|u - v\|_E \leq 1/\sqrt{n}$

(c) $\|\tilde{J}'(v)\|_{E'} \leq 1/\sqrt{n}$.

Taking $u = h_n(s_{n,r})$, we have a sequence $v_n \in E$ such that

(a') $\tilde{J}(v_n) \leq 1/n$

(b') $\|h_n(s_{n,r}) - v_n\|_E \leq 1/\sqrt{n}$

(c') $\|\tilde{J}'(v_n)\|_{E'} \leq 1/\sqrt{n}$.

Notice that (b') implies that $\|v_n\|_E \to r$ as $n \to \infty$. Taking

$$w_n := q + v_n \in \hat{E},$$

the definition of $\tilde{J}(u)$, (a') and (c') imply that

$I(w_n) \to I(q)$ and $\|I'(w_n)\| \to 0$.
as \( n \to \infty \). In addition, each \( w_n \) has the same asymptotics: \( w_n(-\infty) = 0 \) and \( w_n(\infty) = \xi \) for all \( n \). We claim now that there are only two intervals \((-\infty, \bar{t}_{i_1})\) and \((\bar{t}_{n_2}, \infty)\) with unbounded lengths in \( n \) where \( w_n(t) \in B_{\eta/2}(K(V)) \). If not, then there are at least three such intervals, and so by Theorem 1.21, we have

\[
I(q) = \sum_{i=1}^{m} I(x^i)
\]

for some \( m \geq 2 \) and the \( x^i \) are heteroclinic or homoclinic solutions of (HS). Now, at least one of these \( x^i \) connects 0 to \( \xi \). For simplicity, suppose that \( x^1 \) connects 0 to \( \xi \). Thus (101) implies that

\[
0 \geq I(q) - I(x^1) = \sum_{i=1}^{m-1} I(x^i) \geq (m - 1)\nu > 0.
\]

Thus, \( m = 1 \) and so there are only two such intervals. Then, we may apply Proposition 1.13: there exist \( k_n \in \mathbb{Z} \) and a solution \( \tilde{q} \) of (HS) heteroclinic from 0 to \( \xi \) such that

\[
\|\tau_{k_n} w_n - \tilde{q}\|_{W^{1,2}} \to 0
\]

as \( n \to \infty \). By (100), (103) and (b') imply that

\[
\left|\|\tau_{k_n} q - \tilde{q}\|_{W^{1,2}} - r\right| \leq \left|\|\tau_{k_n} q - \tilde{q}\|_{W^{1,2}} - \|\tau_{k_n} v_n\|_{W^{1,2}}\right| + \left|\|\tau_{k_n} v_n\|_{W^{1,2}} - r\right|
\]

\[
\leq \left|\|\tau_{k_n} q - \tilde{q} - (-\tau_{k_n} v_n)\|_{W^{1,2}} + \|v_n\|_{W^{1,2}} - r\right|
\]

\[
\leq \left|\|\tau_{k_n} (q + v_n) - \tilde{q}\|_{W^{1,2}} + \|v_n\|_{W^{1,2}} - r\right| \to 0
\]

as \( n \to \infty \). Thus, \( \|\tau_{k_n} q - \tilde{q}\|_{W^{1,2}} \to r \) as \( n \to \infty \). We claim now that if \( \{k_n\} \subset \mathbb{Z} \) is unbounded, then \( \|\tau_{k_n} q - \tilde{q}\|_{W^{1,2}} \to \infty \). To see this, if \( \{k_n\} \) is not bounded, then passing to a subsequence, we have either \( k_n \to -\infty \) or \( k_n \to \infty \) as \( n \to \infty \). Since \( q(-\infty) = 0 = \tilde{q}(-\infty) \) and \( q(\infty) = \xi = \tilde{q}(\infty) \), there are constants \( \underline{a} < \bar{a} \) such that for \( t < \underline{a}, q(t), \tilde{q}(t) \in B_{1/4}(0) \) and for \( t > \bar{a} \), \( q(t), \tilde{q}(t) \in B_{1/4}(\xi) \). Suppose now that \( k_n \to \infty \). Then, for all \( t > \bar{a}, \tilde{q}(t) \in B_{1/4}(\xi) \). On the other hand, for all \( t < \underline{a} + k_n \), we have \( \tau_{k_n} q(t) \in B_{1/4}(0) \). Thus, for all \( t \in [\bar{a}, \underline{a} + k_n] \), we must have \( \tau_{k_n} q(t) - \tilde{q}(t) \in B_{1/2}(-\xi) \), and so

\[
\|\tau_{k_n} q - \tilde{q}\|_{W^{1,2}}^2 \geq \int_{\underline{a}}^{\bar{a} + k_n} |\tau_{k_n} q(t) - \tilde{q}(t)|^2 dt
\]

\[
\geq (\bar{a} + k_n - \underline{a}) |\xi - 1/2|^2 \to \infty
\]
as \( n \to \infty \). A similar proof holds if \( k_n \to -\infty \) as \( n \to \infty \). Thus, \( k_n = k^* \) for some fixed \( k^* \) and all large \( n \). But then (104) implies that

\[
\|\tau_{k^*} \tilde{q} - \tilde{q}\|_{W^{1,2}} = \|q - \tau_{-k^*} \tilde{q}\|_{W^{1,2}} = r.
\]

Since this is true for any \( r \in (0, \|q(t + 1) - q(t)\|_{W^{1,2}} \), the proof is complete.

In other words, if \( c = 0 \), then there is a sequence \( q_n \) of heteroclinic solutions of (HS) connecting 0 to \( \xi \) such that \( \|q_n - q\|_{W^{1,2}} \to 0 \), which would contradict the assumption that \( q \) is an isolated minimizer of \( I \).

**Theorem 2.2.** If \( c < \nu \) (where \( \nu \) is from Corollary 1.18), then there is a \( u \in E \) such that \( \tilde{J}(u) = c \) and \( \tilde{J}'(u) = 0 \).

Because \( \tilde{J}'(u) = 0 \), we have \( 0 = J_q'(u) \), hence \( u + q \) is a solution of (HS), which goes from 0 to \( \xi \). Moreover, \( I(u + q) - I(q) = c > 0 \), so \( u + q \neq q \), and we have a heteroclinic from 0 to \( \xi \) that is geometrically distinct from \( q \). We shall give an example later of when this theorem is satisfied.

**Proof.** Following [6], we can construct a Palais-Smale sequence \( \{u_n\} \subset E \) such that \( \tilde{J}(u_n) \to c \) and \( \tilde{J}'(u_n) \to 0 \). Using Willem’s notation from [6], we set \( M := [0, 1] \), \( M_0 := \{0, 1\} \). Then our \( \Gamma \) is the same as that in Theorem 2.8 of [6], and since \( c > 0 = \tilde{J}(h(0)) = \tilde{J}(h(1)) \), the assumptions of that theorem are satisfied. Thus, by Theorem 2.9 of [6], there is such a Palais-Smale sequence.

Let \( v_n := u_n + q \). Then, we have \( I(v_n) - I(q) \to c \), hence \( I(v_n) \to c + I(q) \). Notice that we also have \( I'(v_n) \to 0 \), and \( v_n(-\infty) = 0 \), \( v_n(\infty) = \xi \) for all \( n \). By Theorem 1.21, there exists an \( m \in \mathbb{N} \) such that \( c + I(q) = \sum_{i=1}^{m} I(v^i) \), where the \( v^i \) are heteroclinic or homoclinic solutions of (HS). Because \( v^i \) is a chain that connects 0 to \( \xi \), there is at least one \( v^i \) that is heteroclinic from 0 to \( \xi \). For notational convenience, let this heteroclinic be \( v^1 \). We claim now that \( v^1 \) is the only term in \( \sum_{i=1}^{m} I(v^i) \), i.e. \( m = 1 \). If not, we have \( \nu > c > I(v^1) - I(q) + \sum_{i=1}^{m-1} I(v^i) \geq \nu \) since \( I(v^1) - I(q) \geq 0 \) and \( m \geq 2 \). Thus, we must have \( c + I(q) = I(v^1) \). Then, if we define \( u := v^1 - q \), we have

\[
J(u) = J_q(u) - I(q) = I(u + q) - I(q) = I(v^1) - I(q) = c.
\]

Moreover, \( u + q = v^1 \) is a solution to (HS), so \( J'(u) = 0 \)
What if $c \geq \nu$? Then we do not have such precise information as above. However, we have the following

**Theorem 2.3.** Let $q$ be a minimal heteroclinic solution of (HS), connecting 0 to $\xi$, and $p$ be a minimal heteroclinic solution of (HS), connecting $\xi$ to 0. If $c \neq k_1 I(q) + k_2 I(p)$ for $k_1, k_2 \in \mathbb{N}$, $k_1, k_2 \geq 0$ then there is a non-constant $v$ with $I(v) < \infty$, $I(v) = 0$, $v(\pm \infty) \in K(V)$ and $v \not\equiv p, v \not\equiv q$.

**Proof.** Suppose that there is no such $v$. We can find a sequence $\{u_n\} \subset E$ such that $J(u_n) \to c$ and $J'(u_n) \to 0$. If we define $v_n := u_n + q$, we have $J(u_n) = I(u_n + q) = I(v_n) \to c$ and $I'(u_n) \to 0$. Thus, by Theorem 1.21, there is a finite chain $v^1, v^2, \ldots, v^l$ of critical points of $I$ such that $c + I(q) = \sum_{i=1}^l I(v^i)$. Now, by our assumption, each $v^i$ is either $p$ or $q$. Notice that there is at least one $q$ in the sum, since $v^i$ is a chain that connects 0 and $\xi$.

$$c + I(q) = \sum_{j=1}^k I(q) + \sum_{j=1}^{l-k} I(p),$$

hence

$$c = \sum_{j=1}^{k-1} I(q) + \sum_{j=1}^{l-k} I(p)$$

which is a contradiction. \(\square\)

**Remark:**

(1) $v$ is either a heteroclinic, or a homoclinic solution of (HS).

(2) We suspect that the condition $c \neq k_1 I(q) + k_2 I(p)$ is generic, in the sense that it should be possible to find potentials $V_n$ for which this condition is true, and $V_n \to V$ as $n \to \infty$. However, we are not at present able to prove this.

If we know about critical values corresponding to homoclinic solutions, we can sharpen the previous result:

**Corollary 2.4.** Under the conditions of the preceding theorem, if we also know that $c \neq k_1 I(q) + k_2 I(p) + \sum_{\text{finite}} I(h_j)$ where the $h_j$ are homoclinic solutions to (HS), then there is a heteroclinic solution connecting 0 and $\xi$ that is distinct from $p$ and $q$.

We next wish to consider a multiple pendulum type problem, where the potential $V$ is 1-periodic in all of its arguments. More precisely, we assume:

PV1 $V \in C^2(\mathbb{R} \times \mathbb{R}^d, \mathbb{R})$

PV2 $V(t, q_i)$ is 1-periodic in $t, q_i$ for $i = 1, 2, \ldots, d$. 35
PV3 $V(t, \bar{n}) = 0 > V(t, x)$ for all $\bar{n} \in \mathbb{Z}^d, x \notin \mathbb{Z}^d$.

PV4 $\nabla q(t, \bar{n})$ is negative definite for all $\bar{n} \in \mathbb{Z}^d, all t$.

In this case, $K(V) = \mathbb{Z}^d$, and our results from the previous section do not apply to such a potential. However, we can do the following: If $v_n$ is a sequence in $\hat{E}$ such that $I(v_n)$ is bounded and $v_n(-\infty) = 0$ for all $n$, and $\|v_n\|_{L^\infty} \leq M$ for all $n$. Then we modify the potential outside of $B_{M+1}(0) \subset \mathbb{R}^d$ in such a fashion that the new potential agrees with $V$ on the range of the $v_n$, and yet has only finitely many zeros. As it turns out, Lemma 1.3 also applies when $V$ satisfies $PV1 - PV4$. To be more precise, suppose that we have sequence $v_n$ such that

(i) $v_n(-\infty) = 0$ for all $n$

(ii) $I(v_n) < M$

By Lemma 1.3, there is a $C(M)$ such that $\|v_n\|_{L^\infty(\mathbb{R})} \leq C(M)$. Let $\tilde{V}(t, x) := V(t, x) - \phi(x)$, where $\phi \in C^\infty$ is chosen such that $\phi(x) \equiv 0$ for $|x| < C(M) + 1$, $\phi(x) \equiv 1$ for $|x| \geq C(M) + 2$, and $0 \leq \phi \leq 1$. Then, we may apply our results to

$$\tilde{I}(v) := \int_{\mathbb{R}} \left( \frac{1}{2} |\dot{v}|^2 - \tilde{V}(t, v) \right) dt.$$

We identify $\tilde{I}'(v)$ with $\tilde{J}'(0)$, where $\tilde{J}(u) := \tilde{I}(u + v)$.

Let us recall some important facts about $I$, assuming now that $V$ is 1-periodic in all of its arguments. Proofs may be found in [5].

Let $q_1$ be a minimizer of $I$ in $A_1 := \{v \in \hat{E} \mid v(-\infty) = 0 \text{ and } v(\infty) \neq 0\}$. For $2 \leq i \leq n+1$, we inductively define

$$A_i := \{v \in \hat{E} \mid v(-\infty) = 0 \text{ and } v(\infty) \neq \sum_{j=0}^{i-1} k_j q_j(\infty) \text{ where } k_j \in \mathbb{Z}, k_j \geq 0\}$$

where $q_j$ is a minimizer of $I$ over $A_j$. More precisely, we define $A_1$, and choose a minimizer $q_1$ of $I$ over $A_1$. Then we define $A_2$ in terms of $q_1$, and find $q_2$ by minimizing $I$ over $A_2$. Then we define $A_3$ and so on.)

Again, if each $q_i$ is an isolated minimizer of $I$ in $A_i$, then

$$c_i := \inf_{h \in \Gamma_i} \max_{0 \leq s \leq 1} J_i(h_s) > 0,$$

where $J_i(u) := I(q_i + u) - I(q_i)$ and

$$\Gamma_i := \{h \in C([0,1], E) \mid h_0 = 0 \text{ and } h_1(t) = q_i(t + 1) - q_i(t)\}$$
for \( i = 1, 2, \ldots, n+1 \). If \( c_i = 0 \), then we may argue as in Proposition 2.1 to get a contradiction to the assumption that \( q_i \) is an isolated minimizer of \( I \) in \( \mathcal{A}_i \).

Now, we define \( \tilde{V} \) as above with \( M := \max \{ c_i \} + 1 \).

**Proposition 2.5.** Suppose there is a (PS) sequence \( u_n \) of \( J_i \) such that \( J_i(u_n) \to b \) for some \( 0 < b < \nu \). Then there is a \( u \) such that \( J_i(u) = b \) and \( J'_i(u) = 0 \).

**Proof.** Let \( v_n := q_i + u_n \). Then we have \( I(v_n) = J_i(u_n) + I(q_i) \to b + I(q_i) \) and \( I'(v_n) \to 0 \), so \( v_n \) is a (PS) sequence for \( I \). Now, notice that \( v_n(\mathbb{R}) \subset B_{C(M)}(0) \), so \( v_n \) has support in the region where \( \tilde{V} = V \). Notice that this implies \( \tilde{I}(v_n) = I(v_n) \to b + I(q_i) \) and \( \tilde{I}'(v_n) = I'(v_n) \to 0 \), so \( v_n \) is also a (PS) sequence for \( \tilde{I} \). Then by Theorem 1.21, we get a chain of critical points \( \tilde{v}^1, \ldots, \tilde{v}^k \) of \( \tilde{I} \) connecting \( 0 \) to \( q_i(\infty) \). Moreover, we have \( b + I(q_i) = \sum_{j=1}^k \tilde{I}(\tilde{v}^j) \). We claim that each \( \tilde{v}^j \) is also a critical point of \( I \). This follows from the remark following Theorem 1.21: \( \| \tilde{v}^j \|_{L^\infty(\mathbb{R})} < C(M) + 1 \) for \( j = 1, 2, \ldots, k \).

Thus, in fact the \( \tilde{v}^j \) are also critical points of \( I \). Notice then that we have \( b + I(q_i) = \sum_{j=1}^k \tilde{v}^j \), and \( \tilde{v}^1, \tilde{v}^2, \ldots, \tilde{v}^k \) is a chain connecting \( 0 \) and \( q_i(\infty) \). We claim that there is at least one \( \tilde{v}^j \) such that \( \tilde{v}^j - \tilde{v}^j(-\infty) \in \mathcal{A}_i \). If not, then we have the following:

\[
\tilde{v}^1 \text{ connects } 0 \text{ to } \sum_{l=1}^{i-1} k_l^1 q_l(\infty) \tag{106}
\]

Because of the assumption that there is no \( \tilde{v}^j \in \mathcal{A}_i, \tilde{v}^2 - \tilde{v}^2(-\infty) \notin \mathcal{A}_i \). Hence

\[
\tilde{v}^2(\infty) - \tilde{v}^2(-\infty) = \sum_{l=1}^{i-1} k_l^2 q_l(\infty). \tag{107}
\]

But then by (106) and (107)

\[
\tilde{v}^2(\infty) = \tilde{v}^2(-\infty) + \sum_{l=1}^{i-1} k_l^2 q_l(\infty) = \tilde{v}^1(\infty) + \sum_{l=1}^{i-1} k_l^2 q_l(\infty) = \sum_{l=1}^{i-1} (k_l^1 + k_l^2) q_l(\infty).
\]

Continuing on in this spirit, we get that

\[
\tilde{v}^j(\infty) = \sum_{l=1}^{i-1} k_l q_l(\infty) = q_i(\infty),
\]

which contradicts the definition of \( \mathcal{A}_i \). Thus, there is at least one element of the chain that is in \( \mathcal{A}_i \). Suppose that it is \( \tilde{v}^1 \). Notice then that (since \( q_i \) is a minimizer of \( I \) over \( \mathcal{A}_i \) \( I(\tilde{v}^1) = I(q_i) \geq 0 \), and so

\[
\nu \geq b \geq I(\tilde{v}^1) - I(q_i) + \sum_{j=2}^k I(\tilde{v}^j) \geq (k - 1)\nu.
\]

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But then we must have $k = 1$, hence $b = I(\tilde{v}^1) - I(q_i)$. Letting $u := \tilde{v}^1 - q_i$, we have $J_i(u) = I(\tilde{v}^1) - I(q_i) = b$, and since $\tilde{v}^1$ is a critical point of $I$, we have $J'_i(u) = I'(\tilde{v}^1) = 0$. □

**Corollary 2.6.** If $c_i < \nu$ for any $i$, then there is a $u \in E$ such that $J_i(u) = c_i$ and $J'_i(u) = 0$.

**Proof.** We can find a (PS) sequence $u_n$ such that $J_i(u_n) \to c_i$ and $J'_i(u_n) \to 0$. Then, we simply apply the preceding proposition. □

Notice that the Corollary implies that we get a new heteroclinic from 0 to $q_i(\infty)$, namely $v := q_i + u$.

By similar arguments, we get an analog of Theorem 2.3:

**Theorem 2.7.** Suppose that

$$c_i \neq \sum_{finite} I(v^j)$$

where $v^j \in \{q_1, q_2, \ldots, q_{n+1}\} \cup \{\text{new critical values associated with } c_1, c_2, \ldots, c_{i-1}\}$. Then, there must be a non-constant $v$ with $I(v) < \infty$, $I'(v) = 0$, $v \neq q_j$ for any $j$, and $v \neq \text{critical points associated with } c_1, c_2, \ldots, c_{i-1}$.

In other words, if $c_i$ is not a sum of previously known critical values, we must get some new critical point. Again, this new critical point might be a heteroclinic, or a homoclinic. However, if we know that

$$c_i \neq \sum_{finite} I(v^j),$$

where

$$v^j \in \{q_1, q_2, q_{n+1}\} \cup \{\text{new critical values associated with } c_1, c_2, \ldots, c_{i-1}\}$$

$$\cup \{\text{homoclinics}\},$$

then we get a new heteroclinic solution. Notice that we do not specify if this new heteroclinic connects 0 to any of $q_1(\infty), q_2(\infty), \ldots q_{n+1}(\infty)$. We know only that this new heteroclinic solution $\tilde{q}$ of (HS) is an element of the chain connecting 0 to $q_n(\infty)$. Because of the periodicity of $V$, we can translate $\tilde{q}$ such that $\tilde{q}(-\infty) = 0$, but we do not know if $\tilde{q}(\infty)$ equals $q_1(\infty), q_2(\infty), \ldots, q_{n+1}(\infty)$.

How can we verify the assumption that $c < \nu$? An admissible $h \in \Gamma$ is $h_s(t) := q(t + s)$
\( q(t) \). Now, we have

\[
J(h_s) = \int_{\mathbb{R}} \frac{1}{2} |\dot{q}(t+s)|^2 - V(t, q(t+s)) \, dt - \int_{\mathbb{R}} \frac{1}{2} |\dot{q}(t)|^2 - V(t, q(t)) \, dt
\]

\[
= \int_{\mathbb{R}} V(t, q(t)) - V(t, q(t+s)) \, dt
\]

changing variables

\[
= \int_{\mathbb{R}} V(t, q(t)) - V(t - s, q(t)) \, dt
\]

Now, suppose that \( V \) has the form \( V_\varepsilon(t, x) = (1 + \varepsilon g(t)) f(x) \), where \( f, g \) satisfy the following:

(V1)* \( f \) and \( g \) are \( C^2 \), and \( g \) is 1-periodic

(V2)* There is a finite set of points \( K(f) = \{\xi_1, \xi_2, \ldots, \xi_k\} \) such that \( f(x) < 0 = f(\xi_j) \) for all \( x \neq \xi_j \).

(V3)* \( \liminf_{|x|\to\infty} f(x) \leq -\alpha < 0 \)

(V4)* \( f_{qq}(\xi_j) \) is negative definite for \( i = 1, 2, \ldots, k \).

Let \( I_\varepsilon(v) := \int_{\mathbb{R}} \frac{1}{2} |\dot{v}|^2 - V_\varepsilon(t, v) \, dt \). We let \( q_\varepsilon \) be a minimizer of \( I_\varepsilon \). For simplicity, let us assume that \( K(V_\varepsilon) = \{0, \xi\} \) We claim that for \( \varepsilon \) suitably small, \( c < \nu \). Notice that for \( \varepsilon \) suitably small, we have \( \frac{1}{2} \leq 1 + \varepsilon g(t) \leq \frac{3}{2} \). This implies that \( V_\varepsilon(t, x) \) satisfies (V1)-(V4), because \( 1 + \varepsilon g(t) \geq \frac{1}{2} \). However, notice that we would like to have \( \beta_1, \beta_2 \) and \( \alpha(r) \) independent of \( \varepsilon \). This isn’t necessarily going to happen, but we can bound them from below independently of \( \varepsilon \) for \( \varepsilon \) small. For example, we have

\[
-\frac{1}{2} f(x) \leq -(1 + \varepsilon g(t)) f(x) \leq -\frac{3}{2} f(x),
\]

hence \( \beta_1 \) will be bounded below by the smallest eigenvalue of \( -f_{qq}(\xi) \). In much the same fashion, we can bound \( \alpha(r) \) away from 0 independently of \( \varepsilon \). Now, notice that

\[
c \leq \max_{0 \leq s \leq 1} \int_{\mathbb{R}} V_\varepsilon(t, q_\varepsilon(t)) - V_\varepsilon(t - s, q_\varepsilon(t)) \, dt
\]

\[
= \varepsilon \max_{0 \leq s \leq 1} \int_{\mathbb{R}} (g(t) - g(t - s)) f(q_\varepsilon(t))
\]

But \( g(t) - g(t - s) = s g'(\zeta) \), hence \( |g(t) - g(t - s)| \leq s \|g'\|_{L^\infty} \leq \|g'\|_{L^\infty} \). Thus, we have

\[
c \leq \varepsilon \|g'\|_{L^\infty} \int_{\mathbb{R}} -f(q_\varepsilon(t)) \, dt
\]
If we can show that \( \int_{\mathbb{R}} -f(q_\varepsilon) \, dt \) is bounded independent of \( \varepsilon \) small, we'll be done. Recall that we have \( -\frac{1}{2} f(x) \leq - (1 + \varepsilon g(t)) f(x) \leq -\frac{3}{2} f(x) \) for small enough \( \varepsilon \). Thus, we have

\[
\int_{\mathbb{R}} -f(q_\varepsilon) \, dt \leq 2 \int_{\mathbb{R}} -(1 + \varepsilon g(t)) f(q_\varepsilon) \, dt \\
\leq 2 \int_{\mathbb{R}} \frac{1}{2} |\dot{q}_\varepsilon|^2 - (1 + \varepsilon g(t)) f(q_\varepsilon) \, dt \\
= 2I(q_\varepsilon) \\
\leq 2I(\gamma),
\]

where

\[
\gamma(t) := \begin{cases} 
0 & \text{for } t < 0 \\
t\xi & \text{for } 0 \leq t \leq 1 \\
\xi & \text{for } t > 1.
\end{cases}
\]

The last inequality in (108) follows from the fact that \( q_\varepsilon \) is a minimizer of \( I_\varepsilon \) over the class of functions connecting 0 and \( \xi \).

References


