A Constructive Approach to the
Mountain Pass Theorem and Its Relatives

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What is the mountain pass theorem?

**Theorem 1** Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth function such that

(MP1) 0 is a local minimum of $f$

(MP2) There is an $\alpha > 0$ and an $r > 0$ such that $f(x) \geq \alpha$ for all $x$ with

$$\|x\| = r$$

(MP3) There is a $y$ with $f(y) \leq 0$ and $\|y\| \geq r$.

(MP4) $f$ satisfies the Palais-Smale Condition: whenever $f(x_n)$ is bounded and $\nabla f(x_n) \rightarrow 0$, $x_n$ has a convergent subsequence.

Then there is an $\hat{x}$ with $\nabla f(\hat{x}) = 0$ and $f(\hat{x}) \geq \alpha$. 
What does it mean?

(MP1) says $f$ has a valley with a low spot at 0, while (MP2) implies that valley is surrounded by mountains, whose height is at least $\alpha$. (MP3) tells us that there is a point $y$ outside the valley that is at least as low as 0. Thus, there should be a mountain pass - a point $\hat{x} \neq 0$ with $\nabla f(\hat{x}) = 0$. 
What is the role of (MP4)? Let us look at a bad example:

\[ g(x, y) = (y - 2)^2 y^2 + (y - x(y - 1)^2)^2. \]

There is a sequence \( x_n \) such that \( f(x_n) \) is bounded, \( \nabla f(x_n) \to 0 \), and yet \( x_n \) gets arbitrarily large. In addition, this example has two low valleys, separated by a ridge and there are only two critical points!
Suppose we’re looking for a local minimum. How could we find it? Imagine a bead that slides downhill on the landscape. If the elevation on that path is bounded from below, the bead should eventually settle at a low point.
To find a mountain pass, we use a whole string of beads. The first string is a path (any path!) that connects 0 to $y$. We then let the whole string of beads slide downhill for a fixed amount of time, and we get a new path:
We then pick the high-point on this new path, and repeat the process. The string may get longer as we do this! Note also that it is impossible to pull the string off the mountains surrounding the valley, so the elevation of the high points on the string will always be greater than \( \alpha \).
After we do this several times, the high points won’t decrease very much - since if they did, the high points would eventually be below $\alpha$, and the string can’t be pulled off the mountains! Thus, the high points form a sequence $x_n$ with $f(x_n)$ bounded and $\nabla f(x_n)$ going to zero. The Palais-Smale condition then delivers a point $\hat{x}$ where $\nabla f(\hat{x}) = 0$.

Another way of thinking of the process is that as the beads all slide downhill, there will eventually be a spot on the deformed string that can’t be moved anymore - the string is hung up on the landscape. This position where a bead can’t move corresponds to $\nabla f(x) = 0$. 
What about if $n = 3$? It’s a little hard to draw a graph of a function of three variables, so we use color to play the role of the value of $f$.

Using the string, the situation is the same: the string can’t slide off the mountains (the warm ball), so eventually the high points converge to a critical point.
A related situation: the saddle point theorem. Suppose our function $f : \mathbb{R}^3 \to \mathbb{R}$ has a subspace of where it is bounded from below (the red), and a surrounding circle where it is bounded from above (the blue).

Instead of an initial string of beads, we look at a surface of beads. As before, this surface cannot slide off the subspace if sliding decreases $f$!